STRICTLY STRATIFIED ALGEBRAS

István Ágoston[1,2], Vlastimil Dlab[3] and Erzsébet Lukács[1]

Abstract. Generalizing the concept of standardly stratified algebras, we define strictly stratified algebras and show that they are stratified in the sense of Cline, Parshall and Scott. In fact, under certain conditions, the latter two classes of algebras will coincide.

1. Strictly stratified and CPS-stratified algebras

Stratified algebras were recently introduced by Cline, Parshall and Scott in [CPS2]. Independently, as a natural generalization of quasi-hereditary algebras, $\Delta-$filtered algebras were introduced in [D] and the study of these (standardly) stratified algebras advanced in [ADL].

In this note, we introduce a wider class of algebras called strictly stratified or Λ-stratified and show, using an earlier simple characterization of stratified algebras in the sense of Cline, Parshall and Scott (CPS-stratified, for short; see [CPS2]), that strictly stratified algebras are CPS-stratified (Theorem 1.5). For the monomial algebras the concepts of strictly stratified and CPS-stratified algebras coincide (Theorem 2.3). We also show that CPS-stratified algebras of finite global dimension are necessarily quasi-hereditary (Theorem 1.10).

Throughout the paper we shall assume that A is a basic finite dimensional algebra over a field K. The category of finite dimensional right A-modules will be denoted by mod-A. For an idempotent element $e \in A$, $P(e) \simeq eA$ will be the projective right A-module corresponding to e and $S(e)$ will stand for its semisimple top. We shall usually consider an algebra A together with a fixed complete sequence of primitive orthogonal idempotents $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ and write $(A, \mathbf{e})$. In this case we shall use the notation $P(i) = P(e_i)$ and $S(i) = S(e_i)$. Given a sequence $\mathbf{e}$, we define the idempotents $\varepsilon_i = e_i + e_{i+1} + \cdots + e_n$ for $1 \leq i \leq n$ and, for convenience, $\varepsilon_{n+1} = 0$. With this notation we define the *trace filtration* of a module X_A by $0 \subseteq X\varepsilon_n A \subseteq X\varepsilon_{n-1}A \subseteq \cdots \subseteq X\varepsilon_1 A = X$. In particular, the trace filtration of an algebra $(A, \mathbf{e})$ yields the filtration $0 \subseteq A\varepsilon_n A \subseteq A\varepsilon_{n-1}A \subseteq \cdots \subseteq A\varepsilon_1 A = A$ by idempotent ideals.

Recall that for a given $(A, \mathbf{e})$ the *i-th (right) standard module* $\Delta(i)$ is defined by $\Delta(i) = e_i A / e_i A \varepsilon_{i+1} A$. Thus $\Delta(i)$ is the largest factor of $P(i)$ which has no

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composition factors isomorphic to $S(j)$ for $j > i$. Moreover, we define the *i-th (right) proper standard module* $\overline{\Delta}(i)$ by $\overline{\Delta}(i) = e_iA/e_i \operatorname{rad} A\, \varepsilon_i A$; it is the largest factor of $\Delta(i)$ whose radical has no composition factor isomorphic to $S(i)$. Let $\Delta = \{\, \Delta(1), \Delta(2), \dots, \Delta(n)\,\}$ and $\overline{\Delta} = \{\, \overline{\Delta}(1), \overline{\Delta}(2), \dots, \overline{\Delta}(n)\,\}$.

Finally, for a set $\mathcal{C}$ of modules from mod-A, $\mathcal{F}(\mathcal{C})$ will denote the subcategory of modules which are filtered by elements of $\mathcal{C}$. An algebra $(A, \mathbf{e})$ whose right regular representation A_A belongs to $F(\overline{\Delta})$ is called in [ADL] *standardly stratified* (cf. also [CPS2]).

Definition A sequence of (right) A-modules $\Lambda = \big(\Lambda(1), \Lambda(2), \dots, \Lambda(n)\big)$ is called a *(right) stratifying sequence* for $(A, \mathbf{e})$ if it satisfies the following properties:

(*i*) $\Lambda(i)$ is a homomorphic image of the standard module $\Delta(i)$ for $1 \le i \le n$;
(*ii*) $P(i) \in \mathcal{F}\big(\Lambda(i), \Lambda(i+1), \dots, \Lambda(n)\big)$ for $1 \le i \le n$.

The algebra $(A, \mathbf{e})$ is *strictly stratified* or Λ-*stratified* if it has a (right) stratifying sequence $\Lambda = \big(\Lambda(1), \Lambda(2), \dots, \Lambda(n)\big)$.

Recall that the stratified algebras of [ADL] are just those Λ-stratified algebras for which $\Lambda(i)$ equals either $\Delta(i)$ or $\overline{\Delta}(i)$.

To formulate some basic properties of stratifying sequences and strictly stratified algebras, we need the following concept. Let us denote by $\mathcal{P}(e)$ the category of right A-modules which have projective resolutions in terms of *add* eA, i. e. a resolution in which the projective modules are direct sums of direct summands of eA. A standard homological argument gives that a module $X \in$ mod-A is in $\mathcal{P}(e)$ if and only if $\operatorname{Ext}_A^k\big(X, S(1-e)\big) = 0$ for every $k \ge 0$.

Thus, it follows easily that $\mathcal{P}(e)$ is closed under taking directs summands, moreover if two terms of a short exact sequence are in $\mathcal{P}(e)$, then so is the third one. In particular, $\mathcal{P}(e)$ is closed under extensions. We have also the following important lemma.

Lemma 1.1. *If $X \in \mathcal{F}(Y)$ for some $X, Y \in$ mod-A, then $X \in \mathcal{P}(e)$ if and only if $Y \in \mathcal{P}(e)$.*

Proof. Since $\mathcal{P}(e)$ is closed under extensions, $Y \in \mathcal{P}(e)$ and $X \in \mathcal{F}(Y)$ imply $X \in \mathcal{P}(e)$.

Thus, let us assume that $X \in \mathcal{P}(e)$. The condition $X \in \mathcal{F}(Y)$ implies that there exists a short exact sequence

$$0 \to L \to X \to Y \to 0,$$

so that $L \in \mathcal{F}(Y)$. By applying the functor $\operatorname{Hom}_A\big(-, S(1-e)\big)$, the corresponding long exact sequence immediately yields $\operatorname{Hom}_A\big(Y, S(1-e)\big) = 0$. Furthermore, if $\operatorname{Ext}^k\big(Y, S(1-e)\big) = 0$ for some $k \ge 0$, then $L \in \mathcal{F}(Y)$ yields $\operatorname{Ext}^k\big(L, S(1-e)\big) = 0$. Thus, considering the above long exact sequence, we get $\operatorname{Ext}^{k+1}\big(Y, S(1-e)\big) = 0$. Hence, by induction on k, we get that $Y \in \mathcal{P}(e)$. □

Lemma 1.2. *Let $\Lambda = \big(\Lambda(1), \Lambda(2), \dots, \Lambda(n)\big)$ be a stratifying sequence for $(A, \mathbf{e})$. Then:*

(*i*) $\Lambda(i) \in \mathcal{P}(e_i)$ *as $A/A\varepsilon_{i+1}A-$modules;*
(*ii*) $\operatorname{Ext}^k_{A/A\varepsilon_{i+1}A}\big(\Lambda(i), S(j)\big) = 0$ *for every $k \ge 0$ and $i > j$;*
(*iii*) $\operatorname{Ext}^1_A\big(\Lambda(i), S(j)\big) = 0$ *for every $i > j$;*
(*iv*) $\operatorname{Ext}^1_A\big(\Lambda(i), \Lambda(j)\big) = 0$ *for every $i > j$.*

Proof. It is easy to show that for any fixed index i we have the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$. The first implication is clear. Since $A\varepsilon_{i+1}A$ is an idempotent ideal, $\mathrm{Ext}^1\big(\Lambda(i), S(j)\big)$ are the same whether we consider the modules $\Lambda(i)$ and $S(j)$ as A-modules or as $A/A\varepsilon_{i+1}A$-modules (cf. for example [DR]). Hence, (ii) implies (iii). Moreover, (iv) is a consequence of (iii) since the multiplicity $[\Lambda(j) : S(k)] = 0$ for $j < k$. Thus it is enough to show (i) by induction on n.

Clearly, $\Lambda(n) \in \mathcal{P}(e_n)$ follows from $P(n) \in \mathcal{F}\big(\Lambda(n)\big)$ and Lemma 1.1. Hence, by (iv) for $i = n$, we get that the assumption $P(i) \in \mathcal{F}\big(\Lambda(i), \Lambda(i+1), \dots, \Lambda(n)\big)$ is equivalent to the assumption that $e_iAe_nA_A \in \mathcal{F}\big(\Lambda(n)\big)$ and $e_iA/e_iAe_nA \in \mathcal{F}\big(\Lambda(i), \Lambda(i+1), \dots, \Lambda(n-1)\big)$.

Consequently, we may apply induction to the algebra $B = A/Ae_nA$ to obtain the conditions (i)–(iv) for B. Finally, we obtain (iii) and (iv) for A using the isomorphism of the Ext^1_A-modules and Ext^1_B-modules as before. □

Note that Lemma 1.2.(iv) yields a recursive definition for Λ-stratified algebras: $(A, \mathbf{e})$ is $\Lambda = \big(\Lambda(1), \Lambda(2), \dots, \Lambda(n)\big)$-stratified if and only if $e_iAe_nA \in \mathcal{F}\big(\Lambda(n)\big)$ for every i and A/Ae_nA is $\big(\Lambda(1), \Lambda(2), \dots, \Lambda(n-1)\big)$-stratified.

An easy consequence of the previous lemma is the following:

Proposition 1.3 *If $\Lambda = \big(\Lambda(1), \Lambda(2), \dots, \Lambda(n)\big)$ is a stratifying sequence for $(A, \mathbf{e})$, then $A_A \in \mathcal{F}(\Lambda)$ and, furthermore, $\Delta(i) \in \mathcal{F}\big(\Lambda(i)\big)$.*

Proof. The first statement is obvious, while the second one follows from Lemma 1.2 (iv) and an induction argument. Indeed, the condition (iv) of Lemma 1.2 implies readily that every module X from $\mathcal{F}(\Lambda)$ has a Λ-filtration which is a refinement of the trace filtration of X. □

Our next aim is to compare the class of strictly stratified algebras to the class of stratified algebras as defined by Cline, Parshall and Scott.

Following [CPS2] an ideal $I \triangleleft A$ is called a *stratifying ideal* if the following three conditions are satisfied:

- (a) I is an idempotent ideal, hence of the form $I = AeA$ for some $e = e^2 \in A$;
- (b) the multiplication map induces a bijection $Ae \otimes_{eAe} eA \to AeA$;
- (c) $\mathrm{Tor}_i^{eAe}(Ae, eA) = 0$ for $i > 0$.

If there is a chain of ideals $A = I_1 \supseteq I_2 \supseteq \cdots \supseteq I_m \supseteq I_{m+1} = 0$ such that I_i/I_{i+1} is a stratifying ideal in A/I_{i+1} for $1 \le i \le m$, then A will be called CPS-*stratified*. Observe that if A is CPS-stratified, then A^{opp} is also CPS-stratified.

It is easy to see that the existence of such a chain of idempotent ideals is equivalent to having a complete ordered set of primitive orthogonal idempotents $\mathbf{e} = (e_1, e_2, \dots, e_n)$ so that the ideals $A\varepsilon_iA/A\varepsilon_{i+1}A$ are stratifying in $A/A\varepsilon_{i+1}A$ for $1 \le i \le n$. In this case we will refer to the CPS-stratified algebra $(A, \mathbf{e})$.

It is shown in [CPS1] that I is a stratifying ideal if and only if the following condition is satisfied:

- $(*)$ Let $B = A/I$. Then for each pair of modules $X, Y \in$ mod-B we have an isomorphism $\mathrm{Ext}^k_A(X, Y) = \mathrm{Ext}^k_B(X, Y)$ for every $k \ge 0$.

Following [CPS1] and [DR], ideals satisfying the condition $(*)$ were also studied by Auslander, Platzeck and Todorov in [APT]. In particular, we have the following criterion.

Proposition 1.4. ([APT]) *An idempotent ideal $I = AeA$ is a stratifying ideal if and only if I_A belongs to $\mathcal{P}(e)$.*

Proof. We include a proof for the sake of completeness.

First, suppose that AeA is a stratifying ideal. We are going to prove that $AeA_A \in \mathcal{P}(e)$. In view of our earlier remarks, this is equivalent to showing that $\mathrm{Ext}^k\big(AeA, S(1-e)\big) = 0$ for every $k \geq 0$. Since AeA is the trace of the projective module eA in A, the statement is valid for $k = 0$. To prove it for $k > 0$, let us denote by B the factor algebra $B = A\big/AeA$. We have the following exact sequence of (right) A-modules:

$$0 \to AeA \to A \to B \to 0.$$

Since B is projective as a B-module, $\mathrm{Ext}^k_B\big(B, S(1-e)\big) = 0$ for $k > 0$. Thus by the condition $(*)$, also $\mathrm{Ext}^k_A\big(B, S(1-e)\big) = 0$. Finally, since A_A is projective, it turns out that $\mathrm{Ext}^k_A\big(AeA, S(1-e)\big) = 0$ for $k > 0$, as well.

The proof of the converse implication follows — mutatis mutandis — the proof of a similar statement in [DR] for quasi-hereditary algebras and heredity ideals. Thus, let us assume that $AeA_A \in \mathcal{P}(e)$. Then, using the above exact sequence, we get that $\mathrm{Ext}^k_A\big(B, S(1-e)\big) = 0$ for $k > 0$, and thus $\mathrm{Ext}^k_A\big(\overline{P}, S(1-e)\big) = 0$ for every $k > 0$ and for any projective B-module $\overline{P}$.

Now, let us take an arbitrary right B-module X_B and apply the functor $\mathrm{Hom}_A\big(-, S(1-e)\big)$ to the exact sequence

$$0 \to K_B \to \overline{P}_B \to X_B \to 0,$$

where $\overline{P}_B$ is the projective cover of X_B. Then we have $\mathrm{Ext}^k_A\big(X, S(1-e)\big) \simeq \mathrm{Ext}^{k-1}_A\big(K, S(1-e)\big)$, as well as $\mathrm{Ext}^k_B\big(X, S(1-e)\big) \simeq \mathrm{Ext}^{k-1}_B\big(K, S(1-e)\big)$ for arbitrary $k > 0$. Since $\mathrm{Hom}_A(X, Z) \simeq \mathrm{Hom}_B(X, Z)$ for an arbitrary right B-module Z, we get, by induction on k, $\mathrm{Ext}^k_A\big(X, S(1-e)\big) \simeq \mathrm{Ext}^k_B\big(X, S(1-e)\big)$ for $k \geq 0$, i.e. $\mathrm{Ext}^k_A\big(X, Y\big) \simeq \mathrm{Ext}^k_B(X, Y)$ for every simple B-module Y_B. The statement for an arbitrary right B-module Y_B then follows immediately by induction on the length of Y_B.

□

Note that all stratified algebras of [ADL], and in particular, standardly stratified algebras of [ADL] are CPS-stratified.

We may now formulate one of the main results of this section.

Theorem 1.5. *For a given algebra $(A, \mathbf{e})$, let $\Lambda = \big(\Lambda(1), \Lambda(2), \ldots, \Lambda(n)\big)$ be a sequence satisfying $\Delta(i) \in \mathcal{F}\big(\Lambda(i)\big)$ and $A_A \in \mathcal{F}(\Lambda)$. Then $(A, \mathbf{e})$ is CPS-stratified. In particular, all strictly stratified algebras are CPS-stratified.*

Proof. As in the proof of Lemma 1.2, we get easily the following statements: As an $A\big/A\varepsilon_{i+1}A$-module, $\Lambda(i) \in \mathcal{P}(e_i)$, and this implies that there is a Λ-filtration of A_A which is a refinement of the trace filtration of A_A.

Hence $Ae_nA_A \in \mathcal{F}\big(\Lambda(n)\big)$, and since $\Lambda(n) \in \mathcal{P}(e_n)$, we get that $Ae_nA \in \mathcal{P}(e_n)$. Thus Ae_nA is a stratifying ideal. Now, the rest follows by induction on n, since the sequence $\big(\Lambda(1), \Lambda(2), \ldots, \Lambda(n-1)\big)$ satisfies the conditions of the theorem for the algebra $B = A\big/Ae_nA$.

The second statement follows from the first one and Proposition 1.3. □

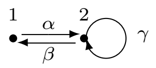
It is worth observing the following simple corollary of the previous theorem and Lemma 1.2.(ii).

Corollary 1.6. *Let $(A, \mathbf{e})$ be a strictly stratified algebra with a stratifying sequence $\Lambda = \big(\Lambda(1), \Lambda(2), \ldots, \Lambda(n)\big)$. Then*

$$\operatorname{Ext}_A^k \big(\Lambda(i), \Lambda(j)\big) = 0 \quad \text{for every } k \geq 0 \text{ and } i > j.$$

The following example shows that a refinement of the trace filtration of A_A to a filtration by general local modules $\big(\Lambda(1), \Lambda(2), \ldots, \Lambda(n)\big)$ (i.e. without assuming that $\Delta(i) \in \mathcal{F}\big(\Lambda(i)\big)$) does not imply that the algebra is CPS-stratified.

Example 1.7. Let A be the factor algebra KQ/I of the path algebra KQ, where

$Q:$  and $I = \langle \alpha\beta, \alpha\gamma\beta, \beta\alpha, \gamma^2 \rangle$.

The algebra has the following right regular representation

$$A_A = \begin{smallmatrix} 1 \\ 2 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 2 \\ & & 1 \end{smallmatrix}.$$

Here, $A_A \in \mathcal{F}\big(S(1), P(2)/P(2)(\operatorname{rad}^2 A)\big)$, but Ae_2A is not a stratifying ideal and thus A is not CPS-stratified.

Finally, we shall prove that CPS-stratified algebras of finite global dimension are necessarily quasi-hereditary (thus generalizing the previous simple characterization of quasi-hereditary algebras by [D] and [W]).

Let us recall that an algebra $(A, \mathbf{e})$ is *quasi-hereditary* if and only if $A_A \in \mathcal{F}(\Delta)$ and $\overline{\Delta}(i) = \Delta(i)$ for every $1 \leq i \leq n$.

Lemma 1.8. *Let $e \in A$ be a primitive idempotent and $X \in \mathcal{P}(e)$. Then proj.dim X is either 0 or ∞. In particular, a stratifying ideal generated by a primitive idempotent is either projective or its projective dimension is ∞.*

Proof. Suppose that $X \in \mathcal{P}(e)$ is not projective and it has a minimal projective resolution of length $d > 0$:

$$0 \to P_d \to P_{d-1} \to \cdots \to P_0 \to X \to 0.$$

Then both P_d and P_{d-1} are direct sums of copies of $P(e)$, so they have the same Loewy-length. This contradicts the fact that P_d is embedded into the radical of P_{d-1}.

□

Theorem 1.9. *The algebra $(A, \mathbf{e})$ is quasi-hereditary if and only if $(A, \mathbf{e})$ is CPS-stratified and gl.dim $A < \infty$.*

Proof. Since quasi-hereditary algebras are CPS-stratified and have finite global dimension, we need only to prove that a CPS-stratified algebra of finite global dimension is necessarily quasi-hereditary.

Let $(A, \mathbf{e})$ be a CPS-stratified algebra of finite global dimension; here, as before, $\mathbf{e} = (e_1, e_2, \ldots, e_n)$. We shall proceed by induction on n. Let $B = A/Ae_nA$. Then, using the characterization $(*)$ of stratifying ideals, we deduce that B is of finite global dimension. Since $(B, (e_1, e_2, \ldots, e_{n-1}))$ is clearly CPS-stratified, it is, by induction hypothesis, quasi-hereditary. Since $gl.dim\, A < \infty$, both Ae_nA_A and ${}_AAe_nA$ are projective by Lemma 1.8. However, since ${}_AAe_nA$ is projective, $Ae_nA_A \in \mathcal{F}\big(\overline{\Delta}(n)\big)$ by a theorem of [D]. Since $\overline{\Delta}(n) \in \mathcal{P}(e_n)$ by Lemma 1.1, $\overline{\Delta}(n)$ is projective by Lemma 1.8 and therefore $\overline{\Delta}(n) = \Delta(n)$. Hence, $(A, \mathbf{e})$ is quasi-hereditary.

2. Existence of local filtrations

In this section we will show that under certain conditions CPS-stratified algebras do have a stratifying sequence.

The first result gives a condition for an algebra to be strictly stratified by the sequence of proper standard modules (hence standardly stratified, i.e. stratified of type $(+1, +1, \ldots, +1)$ in the sense of [ADL]).

THEOREM 2.1. *If $(A, \mathbf{e})$ is CPS-stratified and $\overline{\Delta}(i) \in \mathcal{P}(e_i)$ as an $A/A\varepsilon_{i+1}A$-module for every i, then $(A, \mathbf{e})$ is strictly stratified with a stratifying sequence $\overline{\Delta} = \big(\overline{\Delta}(1), \overline{\Delta}(2), \ldots, \overline{\Delta}(n)\big)$.*

The theorem will follow by applying repeatedly the following proposition.

PROPOSITION 2.2. *For given $(A, \mathbf{e})$ the following are equivalent:*

(*i*) $e_nA = \Delta(n) \in \mathcal{F}\big(\overline{\Delta}(n)\big)$;
(*ii*) $\overline{\Delta}(n) \in \mathcal{P}(e_n)$;
(*iii*) $\mathcal{P}(e_n) = \mathcal{F}\big(\overline{\Delta}(n)\big)$.

Furthermore, if the above conditions are satisfied and Ae_nA is a stratifying ideal then Ae_nA is $\overline{\Delta}(n)$-filtered.

Proof. The implication $(i) \Rightarrow (ii)$ follows from Lemma 1.1 and $(iii) \Rightarrow (i)$ is obvious. Hence we have to prove only $(ii) \Rightarrow (iii)$. Clearly, $\mathcal{F}\big(\overline{\Delta}(n)\big) \subseteq \mathcal{P}(e_n)$; hence, we need to show the opposite inclusion.

We will first prove that for any $X \in \mathcal{P}(e_n)$, $\overline{\Delta}(n)$ is a homomorphic image of X. Let us consider the following exact sequence where $\oplus P(n)$ is the projective cover of X:

$$0 \to K \to \oplus P(n) \to X \to 0.$$

Since $\mathcal{P}(e_n)$ is closed under taking kernels of epimorphisms, $K \in \mathcal{P}(e_n)$ and hence $K \subseteq \oplus\, \mathrm{rad}\, P(n)e_nA$. This means that the canonical surjections $\oplus P(n) \to \overline{\Delta}(n)$ can be factored through X. Hence there is an epimorphism $X \to \overline{\Delta}(n)$ and its kernel X' belongs to $\mathcal{P}(e_n)$.

Now, by induction on the length of X, we get that $X' \in \mathcal{F}\big(\overline{\Delta}(n)\big)$. Hence $\mathcal{P}(e_n) \subseteq \mathcal{F}\big(\overline{\Delta}(n)\big)$.

□

Proof of Theorem 2.1. Since $Ae_nA \in \mathcal{P}(e_n)$, and $\mathcal{P}(e_n)$ is closed under taking direct summands, we get that $e_iAe_nA \in \mathcal{P}(e_n) = \mathcal{F}\big(\overline{\Delta}(n)\big)$ by Proposition 2.2 *(iii)*. Since the conditions of the theorem are clearly inherited for the factor algebra $A\big/Ae_nA$, the result follows by induction on n. □

The next theorem shows that for monomial algebras the converse of Theorem 1.5 holds: here the concepts of strictly stratified and CPS-stratified algebras coincide.

THEOREM 2.3. *A monomial algebra* $(A, \mathbf{e})$ *is CPS-stratified if and only if it is strictly stratified. In particular, a monomial algebra* A *is strictly stratified if and only if* A^{opp} *is strictly stratified.*

In order to prove Theorem 2.3, we need a sequence of statements. First, let us fix some notation.

Throughout the proof, A will denote a finite dimensional monomial K-algebra of the form $A \simeq K\Gamma/I$ with a graph Γ and an ideal I of admissible relations, generated by paths. We shall assume that the vertex set of Γ is $\{\,1, 2, \dots, n\,\}$ and that the set of paths of Γ is written as the disjoint union $\mathcal{P} \,\dot\cup\, \mathcal{P}'$ of the subset $\mathcal{P}'$ of all paths belonging to I and its complement $\mathcal{P}$ in Γ. We will identify $\mathcal{P}$ with a K-basis of A and the primitive orthogonal idempotents $e_1, e_2, \dots, e_n$ with the paths of length 0 at the vertices $1, 2, \dots, n$, respectively. We will multiply paths from left to right: the product of a path p from i to j and a path q from j to k is pq.

For a subset $\Pi \subseteq \mathcal{P}$ we define the *closure* $\overline{\Pi}$ of Π as $\Pi\mathcal{P} \cap \mathcal{P}$. Note that $\overline{\Pi}$ is a K-basis of the right ideal ΠA. Π is *closed* if $\Pi = \overline{\Pi}$. A path $p \in \Pi$ is called *left-minimal in* Π if no proper initial segment of p belongs to Π. For two closed subsets $\Pi' \subseteq \Pi \subseteq \mathcal{P}$ we denote by $M(\Pi/\Pi')$ the factor module $\Pi A\big/\Pi' A$ (or simply $M(\Pi)$ when $\Pi' = \emptyset$). We call a module M a *path-module* if $M \simeq \oplus_t M(\Pi_t/\Pi'_t)$ for some closed subsets $\Pi'_t \subseteq \Pi_t \subseteq \mathcal{P}$. Observe that for $p \in \mathcal{P}$ and a closed subset $\Pi \subseteq \overline{p}$ the module $M(\overline{p}/\Pi)$ is always a local module.

The following lemmas summarize some simple facts about path-modules.

LEMMA 2.4. *Every path module is a direct sum of local path modules:* $M(\Pi/\Pi') \simeq \oplus_{t=1}^{s} M(\overline{p}_t/\Pi'_t)$*, where* p_t *are the left minimal paths in* Π.

Proof. The statement follows easily from the observation that $\Pi = \overline{p}_1 \,\dot\cup\, \overline{p}_2 \,\dot\cup \dots \dot\cup\, \overline{p}_s$, with $p_1, p_2, \dots, p_s$ being the left minimal paths in Π. Then $\overline{p}_t$ and $\Pi'_t = \Pi' \cap \overline{p}_t$ give the required sets of paths for the local direct summands. □

LEMMA 2.5. *If* $\Pi \subseteq \overline{p}$ *is a closed subset for some* $p \in \mathcal{P}$ *with* $pe_i = p$*, then* $M = M(\overline{p}/\Pi) \simeq M(\overline{e}_i/Q)$ *for some* $Q \subseteq \overline{e}_i$.

Proof. Let $f : e_iA \to M(\overline{p}) \subseteq A$ be the surjection defined by $f(q) = pq$ and let g be the natural epimorphism from $M(\overline{p})$ to $M(\overline{p}/\Pi)$. Then $\sum \lambda_i q_i \in \operatorname{Ker} gf$ (with $\lambda_i \in K$) if and only if $\sum \lambda_i pq_i \in M(\Pi)$. Since $q_i \neq q_j$ implies $pq_i \neq pq_j$ or $pq_i = pq_j = 0$, $\sum \lambda_i pq_i \in M(\Pi)$ holds if and only if $pq_i \in \Pi \cup \{\,0\,\}$ for every i with $\lambda_i \neq 0$. Thus $\operatorname{Ker} gf = M(Q)$ with $Q = \{\, q \in \overline{e}_i \,|\, pq \in \Pi \cup \{\,0\,\}\,\}$; note that this is clearly a closed subset of $\overline{e}_i$. This yields $M(\overline{p}/\Pi) \simeq M(\overline{e}_i/Q)$. □

In the next proposition, $\Omega(M)$ denotes the first syzygy of the module M.

Proposition 2.6. *Let M and N be local path modules with $Me_iA = M$ and $Ne_iA = N$. Then they have a common path-module factor L such that the kernels of the surjective maps $M \xrightarrow{f} L$ and $N \xrightarrow{g} L$ are path-modules and $\Omega(L) \in add\{\,\Omega(M), \Omega(N)\,\}$.*

Proof. By Lemma 2.5, $M \simeq M(\overline{e}_i/Q_1)$ and $N \simeq M(\overline{e}_i/Q_2)$ for some closed subsets Q_1 and Q_2 of $\overline{e}_i$. Let $Q = Q_1 \cup Q_2$. Then $L = M(\overline{e}_i/Q)$ is a path-module and it is a homomorphic image of M and N. The kernels of the epimorphisms are isomorphic to $M(Q/Q_1)$ and $M(Q/Q_2)$, respectively. On the other hand $\Omega(L) \simeq QA \simeq \oplus_t \overline{q}_t A$, where $\overline{q}_t A \in add\{\,\Omega(M), \Omega(N)\,\}$ by Lemma 2.4. □

Theorem 2.7. *Let e be a primitive idempotent of the monomial algebra A. Then there is a local module $\Lambda(e)$ such that every path-module of $\mathcal{P}(e)$ is filtered by $\Lambda(e)$. In particular, if AeA is a stratifying ideal, then AeA and its trace on the projective summands of A_A belong to $\mathcal{F}(\Lambda(e))$.*

Proof. Let $M = \Lambda(e)$ be a local path-module of minimal dimension in $\mathcal{P}(e)$. Since $\mathcal{P}(e)$ is closed under taking direct summands, it is enough to prove, in view of Lemma 2.4, that every local path-module N of $\mathcal{P}(e)$ is filtered by M. By Proposition 2.6, M and N have a common factor L, which is a path-module, and its projective cover is $P(e)$. Furthermore, $\Omega(L) \in add\big(\Omega(M), \Omega(N)\big)$ implies that $\Omega(L) \in \mathcal{P}(e)$, and thus $L \in \mathcal{P}(e)$. Now, the minimality of M yields that $M \simeq L$. Since the kernel of the obtained homomorphisms of N onto L is also a path-module in $\mathcal{P}(e)$, we can prove that $N \in \mathcal{F}(M)$ by induction on the length of the module. Finally, if AeA is a stratifying ideal, then AeA and its trace on the projective summands of A_A are in $\mathcal{F}(M)$, since these modules are path-modules. □

Proof of Theorem 2.3. By Theorem 1.5, a strictly stratified algebra is always CPS-stratified. The converse implication follows by induction from Theorem 2.7, using the recursive definition of strictly stratified algebras. The second statement of the theorem follows from the fact that the concept of a CPS-stratified algebra is two-sided. □

Of course, in general not every CPS-stratified algebra is strictly stratified. The following example illustrates such a situation.

Example 2.8. Let A be given by the following right regular representation:

$$A_A = \begin{array}{c}1\\3\\1\\3\end{array} \oplus \begin{array}{ccc} & 2 & \\ & 3 & \\ 1 & & 3\end{array} \oplus \begin{array}{ccc} & 3 & \\ 1 & & \\ 3 & & 3 \\ & 1 & \\ & 3 & \end{array}$$

Clearly, $(A, \mathbf{e})$ is not strictly stratified, but it is CPS-stratified by Proposition 1.4. Observe that A^{opp} is strictly stratified.

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Department of Algebra and Number Theory, Eötvös University, H-1088 Budapest, Hungary
E-mail: agoston@cs.elte.hu

School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, K1S 5B6, Canada
E-mail: vdlab@math.carleton.ca

Department of Algebra, Technical University of Budapest, P.O.Box 91, H-1521 Budapest, Hungary
E-mail: lukacs@math.bme.hu