SUPERPURELY SIMPLE MODULES

by

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§1. Introduction

In Theorem 2.6 of [**DZ**] it is shown that an infinite dimensional torsion-free module of finite rank over a tame hereditary algebra Λ is purely simple if and only if every non-zero homomorphism from M into Q, the unique generic module in the sense of [**CB**], has finite dimensional kernel. In this paper we characterize those torsion-free Λ-modules M for which every non-zero homomorphism $\varphi : M \to Q$ is a monomorphism. We call these modules *superpurely simple*.

From Proposition 6.1.2 in [**R**] it follows that every torsion-free module of rank one is superpurely simple. On the other hand, Theorem 1.6 in [**D**] implies that the infinite dimensional modules of rank larger than one are not superpurely simple. Thus, in order to determine all superpurely simple Λ-modules, one may concentrate on those which are finite dimensional, indecomposable and preprojective of rank larger than one. By first giving a characterization of superpurely simple modules, we show that for each tame hereditary algebra Λ, there are only finitely many non-isomorphic superpurely simple modules of rank larger than one. The precise number of such modules depends on the orientation which is given to the underlying graph of the quiver of Λ. As a consequence of our characterization, we get that for every torsion-free module M of finite rank there exists a chain of torsion-closed submodules $0 = U_0 \subset U_1 \subset \ldots \subset U_t = M$ with U_i/U_{i-1} superpurely simple for $1 \leq i \leq t$. From this we derive a criterion for an infinite dimensional torsion-free module of finite rank to be purely simple.

We recall some basic definitions from [**R**]. Let Λ be a finite dimensional tame hereditary algebra over an algebraically closed field k, that is, a path algebra over a quiver whose underlying graph is one of the Euclidean diagrams $\tilde{A}_n$ $(n \geq 1)$, $\tilde{D}_n$ $(n \geq 4)$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$. We define a torsion theory on the category of right Λ-modules as follows. A module M is called *torsion* if it is spanned by its finite dimensional regular and preinjective submodules. A module M is *torsion-free* if every finite dimensional submodule of M is preprojective. The unique generic module Q constructed by Ringel (cf. Theorem 5.3 of [**R**]) is characterized by the properties that $\operatorname{End} Q$ is a division ring and $\dim_{\operatorname{End} Q} Q < \infty$. Every torsion-free module M can be embedded into a direct sum Y of copies of Q in such a way that the quotient Y/M is *regular* (that is, it has no finite dimensional preprojective or preinjective direct summands) and torsion. The number of copies of Q in a direct sum decomposition of such a module Y is an invariant of M and is called the *rank* of M. We shall denote this number with $\operatorname{rk} M$. By Theorem 1.3 in [**D**], if $\operatorname{rk} M$ is finite then it equals $\dim_{\operatorname{End} Q} \operatorname{Hom}(M, Q)$. The rank is additive on short exact sequences of torsion-free modules (Proposition 2.2 of [**DZ**]).

§2. Basic Properties

A submodule N of a module M is called *torsion-closed* in M if the quotient M/N is torsion-free. The intersection of all torsion-closed submodules in M which contain a given submodule $N \subseteq M$ is torsion-closed in M and it is called the *torsion closure* of N in M. We shall denote it by $\overline{N}^M$ or simply by $\overline{N}$. We shall need the following result.

Proposition 2.1. *Let M be a torsion-free module and $N \subseteq M$ a submodule. Then $\operatorname{rk} \overline{N}^M \leq \operatorname{rk} N$.*

Proof. Assume first that $\operatorname{rk} N < \infty$ and consider the following exact sequence:

$$0 \longrightarrow N \xrightarrow{\varphi} \overline{N} \xrightarrow{\psi} \overline{N}/N \longrightarrow 0.$$

Applying the functor $\operatorname{Hom}(-, Q)$ we get:

$$0 \longrightarrow \operatorname{Hom}(\overline{N}/N, Q) \xrightarrow{\psi^*} \operatorname{Hom}(\overline{N}, Q) \xrightarrow{\varphi^*} \operatorname{Hom}(N, Q).$$

Since $\overline{N}/N$ is torsion, $\operatorname{Hom}(\overline{N}/N, Q) = 0$. Thus φ^* is a monomorphism and so $\dim_{\operatorname{End} Q} \operatorname{Hom}(\overline{N}, Q) \leq \dim_{\operatorname{End} Q} \operatorname{Hom}(N, Q)$. But $\dim_{\operatorname{End} Q} \operatorname{Hom}(N, Q) = \operatorname{rk} N$ and the finiteness of this implies that also $\dim_{\operatorname{End} Q} \operatorname{Hom}(\overline{N}, Q) = \operatorname{rk} \overline{N}$. Thus $\operatorname{rk} \overline{N} \leq \operatorname{rk} N$, as required.

Let us consider now the case when $\operatorname{rk} N = \kappa \geq \aleph_0$. Suppose that $\operatorname{rk} \overline{N}^M = \lambda > \kappa$. From Lemma 4.2 and Proposition 4.3 in [**R**] we get that there exists a submodule $\bigoplus_{i \in I} P_i \leq N$ such that P_i is finite dimensional preprojective for each $i \in I$ and $N / \bigoplus_{i \in I} P_i$ is torsion regular. Since for every $i \in I$ we have that $\operatorname{rk} P_i < \aleph_0$ and since the class of torsion regular modules is closed under extensions (cf. Proposition 4.2 in **[R]**), we get that $\operatorname{rk} N = |I| = \kappa$. Consider now the following embeddings:

$$\bigoplus_{i\in I} P_i \hookrightarrow N \hookrightarrow \overline{N}^M \hookrightarrow \bigoplus_J Q = Y$$

with $|J| = \lambda = \operatorname{rk} \overline{N}$ and the quotient module $Y/\overline{N}^M$ torsion regular. Let us remark first that since the modules $N/\bigoplus_{i\in I} P_i$, $\overline{N}^M/N$ and $Y/\overline{N}^M$ are all torsion, we get that $\overline{\bigoplus_{i\in I} P_i}^Y = Y$. We show that this leads to a contradiction.

Since $\dim_k P_i < \aleph_0$ for each $i \in I$, we obtain that $\kappa \leq \dim_k \bigoplus_{i\in I} P_i \leq \kappa \cdot \aleph_0 = \kappa$. This also gives that the set $J' = \{j \in J \mid \pi_j(\bigoplus_{i\in I} P_i) \neq 0\}$, where $\pi_j : Y \to Q$ is the j'th projection, satisfies $|J'| \leq (\dim_k \bigoplus_{i\in I} P_i) \cdot \aleph_0 = \kappa < \lambda = |J|$. Hence $J \setminus J'$ is not empty, that is, there is an index $j \in J$ such that $\operatorname{Ker} \pi_j \supseteq \bigoplus_{i\in I} P_i$. But then for the torsion closure of $\bigoplus_{i\in I} P_i$ we get $\overline{\bigoplus_{i\in I} P_i}^Y \subseteq \operatorname{Ker} \pi_j \subset Y$ since $\operatorname{Ker} \pi_j$ is obviously torsion-closed, and this is a contradiction. Thus $\operatorname{rk} \overline{N}^M \leq \operatorname{rk} N$.

Definition 2.2. A torsion-free module M is called *superpurely simple* if for every non-zero homomorphism $\varphi : M \to Q$ we have $\operatorname{Ker} \varphi = 0$.

Superpurely simple modules obviously must be indecomposable. The following characterization will be used in §3 for determining all superpurely simple modules.

Theorem 2.3. *Let M be a torsion-free module. The following statements are equivalent.*

(i) M is not superpurely simple.

(ii) There exists a non-zero torsion-closed submodule $U \subset M$.

(iii) There exists a non-zero submodule $U \subset M$ with $\operatorname{rk} U < \operatorname{rk} M$.

(iv) There exists a non-zero homomorphism $\varphi : V \to M$ from a torsion-free module V with $\operatorname{rk} V < \operatorname{rk} M$.

Proof. $(i) \Rightarrow (ii)$ Let $\varphi : M \to Q$ be a non-zero homomorphism with non-zero kernel. Let $U = \operatorname{Ker} \varphi$. Then U is torsion-closed and $U \neq M$.

$(ii) \Rightarrow (iii)$ Assume first that $\operatorname{rk} M < \infty$. Consider the sequence

$$0 \longrightarrow U \longrightarrow M \longrightarrow M/U \longrightarrow 0$$

with $0 \neq U \subset M$ and U torsion-closed in M. Then M/U is non-zero and torsion-free so we have $\operatorname{rk} M = \operatorname{rk} U + \operatorname{rk} M/U$. Since $\operatorname{rk} M/U > 0$, we get $\operatorname{rk} U < \operatorname{rk} M$, as required.

To show the statement for the case when $\operatorname{rk} M = \infty$, we will show that every torsion-free module contains a torsion-closed submodule U of finite rank. Let $W \leq M$ be a finite dimensional submodule. Clearly $\operatorname{rk} W < \infty$. By Proposition 2.1, with $U = \overline{W}$, we get that $\operatorname{rk} U \leq \operatorname{rk} W < \infty$. Hence U is torsion-closed of finite rank.

$(iii) \Rightarrow (iv)$ Choose the embedding of U into M.

$(iv) \Rightarrow (iii)$ Take $U = \operatorname{Im} \varphi \leq M$. Then $\operatorname{Im} \varphi \neq 0$ and $\operatorname{rk} \operatorname{Im} \varphi = \operatorname{rk} V - \operatorname{rk} \operatorname{Ker} \varphi \leq \operatorname{rk} V < \operatorname{rk} M$.

$(iii) \Rightarrow (ii)$ By Proposition 2.1, if $\operatorname{rk} U < \operatorname{rk} M$, then $\overline{U} < M$, hence M has a proper torsion-closed submodule, as required.

$(ii) \Rightarrow (i)$ For $U \subset M$, where U is torsion-closed in M, the module M/U is torsion-free, hence we have a non-zero homomorphism $\varphi : M/U \to Q$. By combining it with the natural epimorphism $M \to M/U$, we get a non-zero homomorphism $\tilde{\varphi} : M \to Q$ with $U \subseteq \operatorname{Ker} \tilde{\varphi}$. Since by assumption $U \neq 0$, we have $\operatorname{Ker} \tilde{\varphi} \neq 0$, as required.

Corollary 2.4. *Every non-zero torsion-free module M contains a non-zero superpurely simple torsion-closed submodule.*

Proof. Choose a non-zero torsion-closed submodule U of minimal rank. The argument in the proof of Theorem 2.3 $(ii) \Rightarrow (iii)$ shows that $\operatorname{rk} U < \infty$. Then any proper torsion-closed submodule of U would be torsion-closed in M and of smaller rank than U. This would contradict the minimality of $\operatorname{rk} U$. Hence by (ii) U is superpurely simple.

Corollary 2.5. *Let M be a torsion-free module of finite rank. Then there exists a chain $0 = U_0 \subset U_1 \subset \ldots \subset U_t = M$ of torsion-closed submodules in M such that U_i/U_{i-1} is superpurely simple for $i = 1, \ldots t$.*

Proof. Choose a non-zero superpurely simple torsion-closed submodule $U_1 \leq M$, given by Corollary 2.4. Then either $M = U_1$, hence M is superpurely simple, or $\operatorname{rk} M/U_1 < \operatorname{rk} M$, and in this case an induction argument completes the proof.

At the end of §3 we will see an example showing that neither the isomorphism types of the consecutive quotients, nor the number of the elements in the chain of Corollary 2.5 are uniquely determined by M (see Example 3.4). However, for purely simple modules the following holds. (Recall that a submodule $N \subseteq M$ is called *pure* in M if N is a direct summand in every submodule $U \subseteq M$ containing N for which U/N has finite length. A module M is *purely simple* if it has no non-zero proper pure submodules.)

Proposition 2.6. *Let M be an infinite dimensional torsion-free module of finite rank. Then M is purely simple if and only if for every chain $0 = U_0 \subset U_1 \subset \ldots \subset U_{t-1} \subset U_t = M$ of torsion-closed submodules with U_i/U_{i-1} superpurely simple for $1 \leq i \leq t$, we have $\dim_k U_{t-1} < \infty$.*

Proof. Let M be an infinite dimensional torsion-free module. If M is purely simple of finite rank, then by Theorem 2.4 of [**DZ**] every proper torsion-closed submodule must be finite dimensional. Hence $\dim_k U_{t-1} < \infty$. If M is not purely simple then by Theorem 2.6 of [**DZ**] there is an infinite dimensional proper torsion-closed submodule $U \subset M$. If we take a chain of torsion-closed submodules of U with superpurely simple quotients and complete it to a chain of M as follows:

$$0 = U_0 \subset U_1 \subset \ldots \subset U_i = U \subset U_{i+1} \subset \ldots \subset U_t = M,$$

then $\dim_k U_i$ and hence $\dim_k U_{t-1}$ are infinite.

Thus we get the following corollary (see also Proposition 1.2 in [**O2**]). The proof is obvious.

Corollary 2.7. *Let M be a purely simple infinite dimensional torsion-free module of finite rank and let N be a torsion-closed submodule of M. Then M/N is also purely simple.*

It would be interesting to know if Corollary 2.7 holds if M is of infinite rank. Actually the existence or non-existence of purely simple modules of infinite rank is still an open question (see [**O1**]).

For torsion-free regular modules Corollary 2.5 can be slightly strengthened.

Proposition 2.8. *Let M be a torsion-free regular module of finite rank t. Then there exists a chain of torsion-closed submodules $0 = U_0 \subset U_1 \subset \ldots \subset U_{t-1} \subset U_t = M$ such that $\operatorname{rk} U_i/U_{i-1} = 1$ for $1 \leq i \leq t$.*

Proof. Let P be an indecomposable projective module of rank one. By Proposition 4.3 in [**R**] we can find a submodule $U = \bigoplus_{i \in I} P_i \subseteq M$ such that $P_i \cong P$ for $i \in I$ and M/U

is torsion regular. Since the class of torsion regular modules is closed under extensions (cf. Proposition 4.2 in [**R**]) we get that $\operatorname{rk} U = \operatorname{rk} M = t$, thus we may assume $I = \{1, \ldots, t\}$. Take $U_j = \overset{j}{\underset{i=1}{\oplus}} P_i$. The proof of Proposition 4.3 in [**R**] actually shows that $U_j \cap P_{j+1} = 0$ for $1 \leq j \leq t-1$. Then Proposition 2.1 implies that $\operatorname{rk} U_j / U_{j-1} = 1$, as required.

§3. Superpurely simple modules in the Auslander–Reiten quiver

In this section we use Theorem 2.3 to examine in more detail the distribution of superpurely simple modules in the Auslander–Reiten quiver of Λ. In what follows, $\tau P = \mathrm{DTr}\, P$ denotes the Auslander–Reiten translate of the finite dimensional indecomposable module P. Let us also recall that for a finite dimensional preprojective module P the rank of P equals the negative of the defect of P (Proposition 5.6 in [**R**]). Since the defect is invariant under τ, we have $\operatorname{rk} P = \operatorname{rk} \tau P$.

Proposition 3.1. *Let P be a (finite dimensional) indecomposable preprojective module which is not projective. If P is superpurely simple, then so is τP.*

Proof. Assume that τP is not superpurely simple. Then by (iv) of Theorem 2.3 we can find a non-zero homomorphism $\varphi : P' \to \tau P$ from an indecomposable preprojective module P' for which $\operatorname{rk} P' < \operatorname{rk} \tau P$. But then $\tau^{-1}\varphi : \tau^{-1} P' \to P$ is also non-zero. Since $\operatorname{rk} \tau^{-1} P' = \operatorname{rk} P' < \operatorname{rk} \tau P = \operatorname{rk} P$, condition (iv) of Proposition 2.3 implies that P is not superpurely simple.

Let Γ be a Euclidean diagram with vertices $\{1, \ldots, n+1\}$ and with an orientation $\vec{\Gamma}$. Let $P(i)$ denote, for $1 \leq i \leq n+1$, the indecomposable projective module corresponding to the vertex i over the path algebra $k\vec{\Gamma}$. Then the rank of $P(i)$ is actually independent of the particular orientation $\vec{\Gamma}$ (see for example section 1.D of [**R**]). Thus the number of indecomposable projective modules of rank one over $k\vec{\Gamma}$ (which is also independent of the orinetation) will be denoted by $s(\Gamma)$.

Proposition 3.2. *Let Γ be a Euclidean diagram on $n+1$ vertices. Then for every natural number t such that $s(\Gamma) \leq t \leq n+1$ there exists an orientation $\vec{\Gamma}$ of Γ such that the number of non-isomorphic indecomposable projective $k\vec{\Gamma}$-modules which are superpurely simple equals t.*

Proof. Since the algebra $k\vec{\Gamma}$ is hereditary, condition (iv) of Theorem 2.3 implies that the projective modules $P(j)$ is not superpurely simple if and only if there exists a non-zero homomorphism $\varphi : P(i) \to P(j)$ for some i where $\operatorname{rk} P(i) < \operatorname{rk} P(j)$. It is also clear that there is a non-zero homomorphism $\varphi : P(i) \to P(j)$ if and only if there is an oriented path in $\vec{\Gamma}$ from j to i. Thus $P(j)$ is superpurely simple if and only if there is no oriented path in $\vec{\Gamma}$ from j to any i with $\operatorname{rk} P(i) < \operatorname{rk} P(j)$. As $\operatorname{rk} P(i)$ is independent of the orientation of Γ, the statement follows by an easy case-by-case analysis, or from Proposition 3.5 below.

Theorem 3.3. *Let Γ be a Euclidean diagram. Then the number of non-isomorphic superpurely simple $k\vec{\Gamma}$-modules of rank larger than one is finite for any orientation $\vec{\Gamma}$ of Γ.*

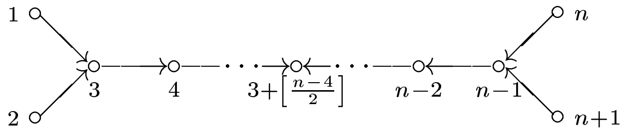
Proof. As pointed out earlier, any superpurely simple module of rank larger than one must be finite dimensional, indecomposable and preprojective. (Note that for $\Gamma = \tilde{A}_n$ all finite dimensional indecomposable preprojective modules are of rank one.) So let us assume that i is a vertex with $\operatorname{rk} P(i) > 1$ and let t be the distance of the vertex i in the graph Γ from a vertex j with $\operatorname{rk} P(j) < \operatorname{rk} P(i)$. Then regardless of the orientation of Γ, condition (iv) of Theorem 2.3 implies that $\tau^{-t}P(i)$ is not superpurely simple. Hence, by Proposition 3.1, each τ-orbit containing an indecomposable projective of rank larger than one has only finitely many superpurely simple modules.

The following is the list of the maximal number of superpurely simple modules of rank larger than one corresponding to some orientation of the Euclidean diagrams. The numbers, denoted by $m(\Gamma)$ can be checked by a simple case-by-case analysis.

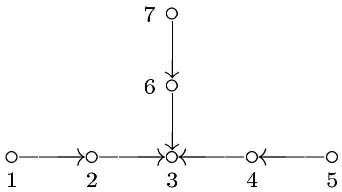
Γ	$m(\Gamma)$
$\tilde{A}_n$	0
$\tilde{D}_n$	$\left[\frac{n-2}{2}\right] \cdot \left[\frac{n-1}{2}\right]$
$\tilde{E}_6$	4
$\tilde{E}_7$	8
$\tilde{E}_8$	13

The following orientations give a maximal number of superpurely simple modules of rank larger than one.

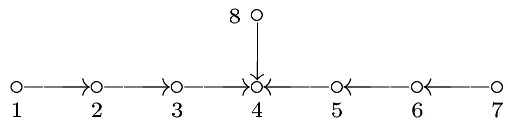
$\tilde{D}_n$:



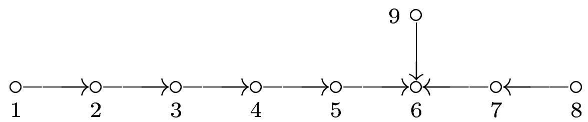
$\tilde{E}_6$:



$\tilde{E}_7$:

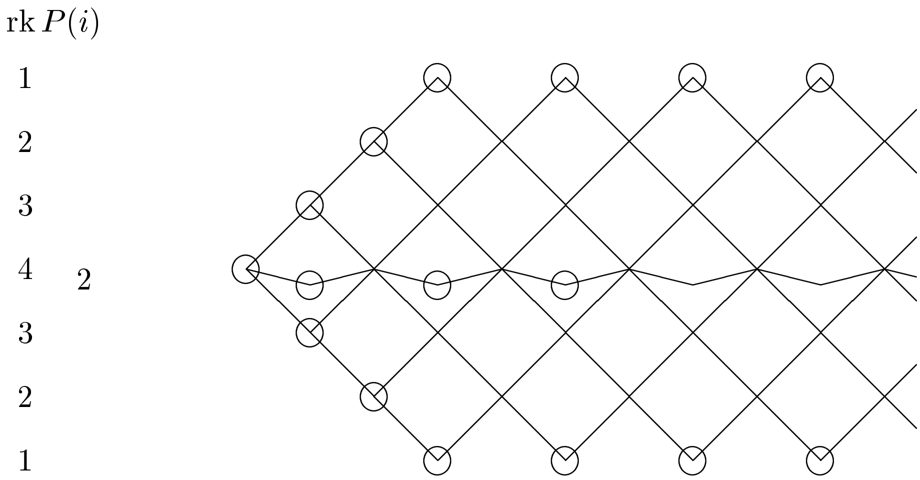


$\tilde{E}_8$:



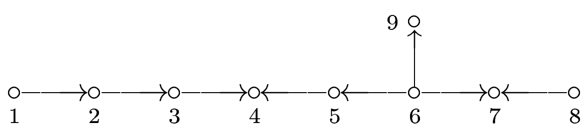
As an illustration, we give the position of the superpurely simple modules in the Auslander–Reiten quiver of $k\vec{\Gamma}$ for $\Gamma = \tilde{E}_7$ with the orientation given above. The circled vertices correspond to the superpurely simple modules.

$\tilde{E}_7$:



Example 3.4. The following example shows that for chains of torsion closed submodules with superpurely simple quotients, neither the isomorphism types of the quotients, nor the number of the elements in these chains is uniquely determined by the module.

Consider the graph $\Gamma = \tilde{E}_8$ with the following orientation:



On Figure 1 we show a portion of the preprojective component of the Auslander–Reiten quiver of the corresponding path algebra $k\vec{\Gamma}$.

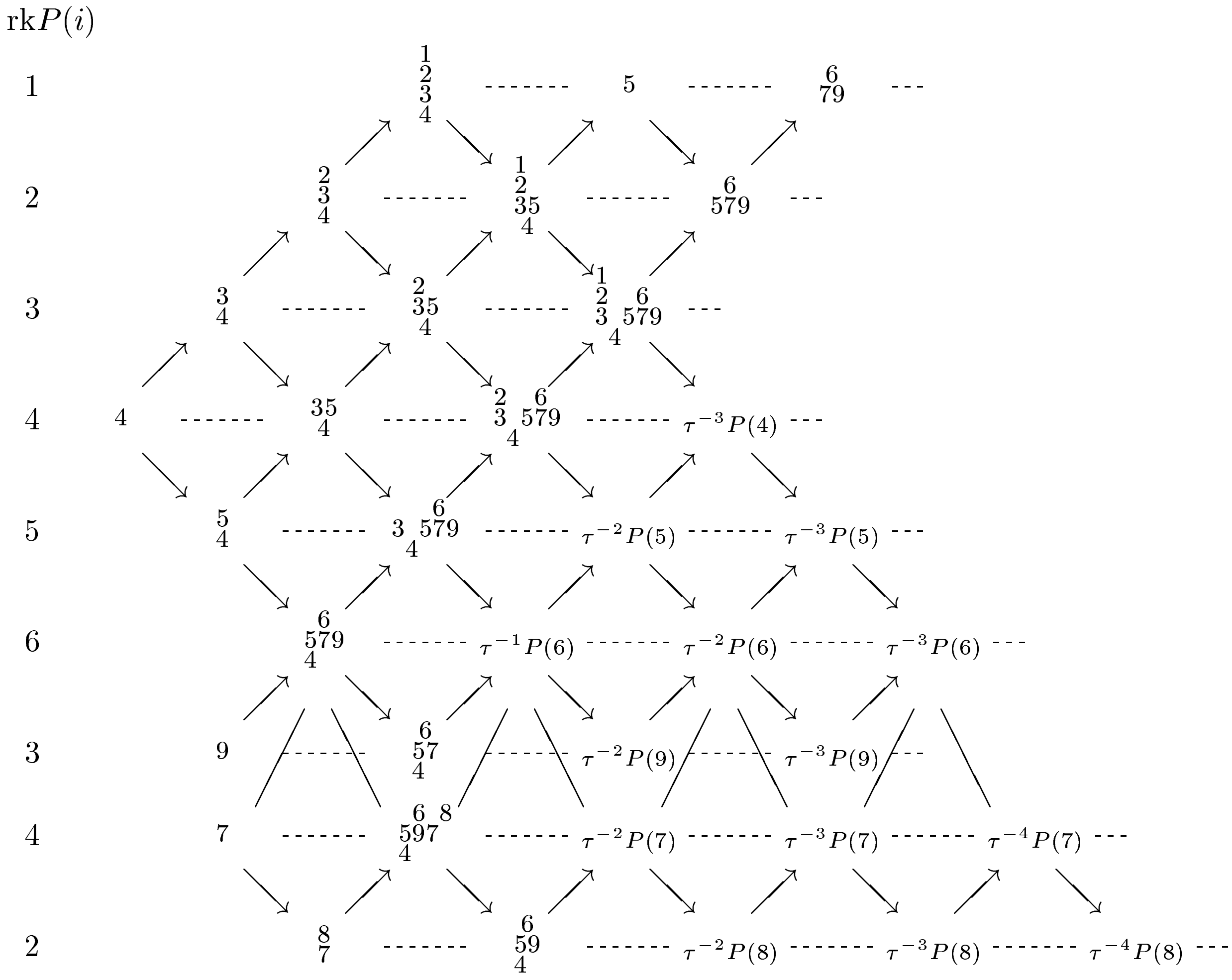
The projective module $P(6)$ has three different filtrations by torsion-closed submodules:

(i) $0 \subset P(4) \subset P(5) \subset P(6)$; here $P(5)/P(4) = \tau^{-1}P(1)$ and $P(6)/P(5) = \tau^{-2}P(1)$, thus $\operatorname{rk} P(4) = 4$, $\operatorname{rk} P(5)/P(4) = \operatorname{rk} \tau^{-1}P(1) = 1$ and $\operatorname{rk} P(6)/P(5) = \operatorname{rk} \tau^{-2}P(1) = 1$.

(ii) $0 \subset P(7) \subset P(6)$; here $\operatorname{rk} P(7) = 4$, $\operatorname{rk} P(6)/P(7) = \operatorname{rk} \tau^{-1}P(8) = 2$.

(iii) $0 \subset P(9) \subset P(6)$; here $\operatorname{rk} P(9) = 3$, $\operatorname{rk} P(6)/P(9) = \operatorname{rk} \tau^{-1}P(9) = 3$.

The quotients considered above are all superpurely simple as can be easily checked on the Auslander–Reiten quiver of Figure 1. (The modules at the beginning of the quiver are represented by their Loewy series.)



○ superpurely simple modules

Figure 1

We finish our paper with a purely combinatorial proposition. It may be used as an alternate proof to the case-by-case checking in Proposition 3.2. Let $\Gamma = (V, E)$ denote an unoriented graph with V being the set of vertices and E the set of edges. Let $v : V \to \{1, 2, \ldots, \ell\}$ be an arbitrary function. A vertex $v \in V$ is said to be *superpurely simple* for a given orientation $\vec{\Gamma}$ of Γ if there is no path in $\vec{\Gamma}$ from g to some vertex $h \in V$ with $v(h) < v(g)$.

Proposition 3.5. [**L**] *Let $\Gamma = (V, E)$ be a finite connected unoriented graph with a function $v : V \to \{1, 2, \ldots, \ell\}$. Then for any natural number m satisfying $|\{g \in V \mid \forall h \in V\ v(g) \leq v(h)\}| \leq m \leq |V|$ there is an orientation $\vec{\Gamma}$ of Γ such that the number of superpurely simple vertices with respect to this orientation is exactly m.*

Proof. We will need the following lemma.

Lemma 3.6. *Let $\Gamma = (V, E)$ and v be as above. Let $A = \{g \in V \mid v(g) < t\}$ and $B = \{h \in V \mid v(h) \geq t\}$ for some (fixed) value t. Assume that for every $h \in B$ there is a path leading from h to some vertex $g \in A$. Then there is an orientation $\vec{\Gamma}$ of Γ such that the vertices which are superpurely simple with respect to this orientation are exactly the elements of A.*

Proof. Choose an orientation according to the following rules:

(a) $g, h \in A$, $g \to h$ implies $v(g) \leq v(h)$;

(b) $g \in A$, $h \in B$ implies that if there is an edge between g and h, then $h \to g$;

(c) $g, h \in B$, $g \to h$ implies that $d_A(g) \geq d_A(h)$, where $d_A(g)$ and $d_A(h)$ denote the distances of g and h from A.

This orientation clearly satisfies the requirements.

Let us turn now to the proof of the proposition. Fix the value m satisfying the given conditions. Then we can choose t to be such that $|\{g \in V \mid v(g) < t\}| < m \leq |\{g \in V \mid v(g) \leq t\}|$. Finally, let us define the following subsets of V:

$$\begin{aligned}
A &= \{g \in V \mid v(g) < t\}; \\
B &= \{g \in V \mid v(g) = t\}; \\
B_i' &= \{g \in B \mid \text{every path connecting } g \text{ with an element of } A \text{ contains at least } i \text{ elements} \\
&\qquad \text{of } B, \text{ counting } g \text{ too}\};
\end{aligned}$$

$$B_i \;=\; B'_i \setminus B'_{i+1}.$$

Thus B is the disjoint union of the sets $B_1, B_2, \ldots, B_s$, and from an element of B_i there is always a path to an element of A not containing any of the elements of $B_{i+1}, \ldots, B_s$. Choose now a subset $C \subseteq B$ such that:

(i) $|A \cup C| = m$;

(ii) if $C \cap B_i \neq \emptyset$, then $C \supseteq \bigcup_{j=i+1}^{s} B_j$.

Split the set of vertices V into the following two disjoint sets:

$$V_1 = \{g \in V \setminus C \mid \text{there is a path from } g \text{ to } A \text{ in } \Gamma \setminus C\};$$

$$V_2 = V \setminus V_1.$$

Let us first note that clearly $A \cup (B \setminus C) \subseteq V_1$, and $C \subseteq V_2$. Let G_1 and G_2 denote the subgraphs of G on the vertices V_1 and V_2, respectively. Apply now Lemma 3.6 for G_1 and G_2 with decompositions $V_1 = A \cup (V_1 \setminus A)$ and $V_2 = C \cup (V_2 \setminus C)$. Both decompositions satisfy the requirements. Complete the so found orientations of G_1 and G_2 to an orientation of G by taking $g \to h$ if $g \in V_1$, $h \in V_2$ and there is an edge between g and h. Then the set of superpurely simple vertices is $A \cup C$, since orienting the edges from V_1 to V_2 will preserve this property both for vertices in A and in C.

Acknowledgement

This paper was written while the first author was visiting Bishop's University. He wishes to express his gratitude to Prof. Andrew Dean for his warm hospitality. The financial support of the Natural Sciences and Engineering Research Council of Canada, of the Alexander von Humboldt Foundation, of Carleton University and of Université de Sherbrooke is gratefully acknowledged.

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