STANDARDLY STRATIFIED EXTENSION ALGEBRAS

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Abstract. The paper generalizes some of our previous results on quasi-hereditary Koszul algebras to graded standardly stratified Koszul algebras. The main result states that the class of standardly stratified algebras for which the left standard modules as well as the right proper standard modules possess a linear projective resolution — the so called linearly stratified algebras — is closed under forming their Yoneda extension algebras. This is proved using the technique of Hilbert and Poincaré series of the corresponding modules.

Introduction

In our earlier paper [ADL4] we have identified a class of quasi-hereditary algebras which is closed with respect to forming the Yoneda extension algebra. As a consequence, all algebras associated with the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ of [BGG] were shown to belong to that class. One of the objectives of the present paper is to extend these results to properly stratified algebras, a class of algebras which has drawn much attention lately (cf. [D2], [FM], [FKM], [KKM], [M]).

Recall the definition of *standardly stratified algebras* $(A, \mathbf{e})$ as those whose left regular representation ${}_AA$ is filtered by standard modules $\Delta^\circ(i)$ (or, equivalently by [D1], whose right projective representation A_A is filtered by proper standard modules $\overline{\Delta}(i)$). An algebra A is called *properly stratified* if both A and A^{opp} are standardly stratified. An algebra will be called *linearly stratified* if it is a standardly stratified Koszul algebra whose right proper standard modules $\overline{\Delta}(i)$ and left standard modules $\Delta^\circ(i)$ all have linear (top) projective resolutions. The main result of the present paper concerns tightly graded algebras (i. e. positively graded algebras $A = \oplus_{r\geq 0} A_r$ generated in degrees 0 and 1 with finite dimensional A_r and semisimple A_0) and can be stated as follows.

Theorem. *A tightly graded algebra $(A, \mathbf{e})$ is linearly stratified if and only if its Yoneda extension algebra (i. e. its homological dual) is linearly stratified.*

1991 Mathematics Subject Classification. Primary 16E30.

[1] Research partially supported by Hungarian NFSR grants no. T034350 and T043034

[2] Part of this work was done while the author held the Széchenyi Professor's Fellowship

[3] Research partially supported by NSERC of Canada grant no. A-7257

[4] Part of this work was done while the author was a visitor at the Paul Erdős Research Centre

Linearly stratified algebras

Throughout the paper, $A = \bigoplus_{r\geq 0} A_r$ will denote a basic *positively graded* connected algebra over a field K, with $dim_K A_r < \infty$ for all r. Unless otherwise stated, we shall also assume that A is *tightly graded.* Thus A_0 is a finite dimensional semisimple algebra and $A_r \cdot A_s = A_{r+s}$ for all integers $r, s \geq 0$. Obviously the (graded) radical of A is $\operatorname{rad} A = \bigoplus_{r\geq 1} A_r$. Let us fix a primitive orthogonal decomposition of the identity element $1 = e_1 + e_2 + \ldots + e_n$ so that $e_i \in A_0$ and keep the order $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ of this complete set of primitive orthogonal idempotents throughout the paper. By a *graded (right) A-module* X we shall always mean a vector space $X = \bigoplus_{r\geq k} X_r$ for some $k \in \mathbb{Z}$ with $X_r \cdot A_s \subseteq X_{r+s}$ where all X_r are finite dimensional. The module X is said to be *generated in degree k* if $X = X_k \cdot A$ (thus $X_k \cdot A_r = X_{k+r}$ for every $r \geq 0$). Note that in this case $X_r = 0$ for all $r < k$. A submodule Y of X is said to be a *graded submodule* of X if $Y = \bigoplus_{r\geq k} Y_r$ with $Y_r = Y \cap X_r$. If both X and Y are generated in the same degree k then Y is called a *top submodule* of X. If $X = \bigoplus_{r\geq k} X_r$ is a graded module generated in degree k, then the graded submodule $\operatorname{rad}^t X = \bigoplus_{r\geq t+k} X_r$ is generated in degree $t + k$ for all $t > 0$.

Note that the right regular representation A_A and its indecomposable direct summands $P(i) = e_i A$ are graded modules generated in degree 0 and the graded submodules $\operatorname{rad}^t A_A$ and $\operatorname{rad}^t P(i)$ in degree $t \geq 1$. Similarly, the irreducible A-modules $S(i) = P(i)/\operatorname{rad} P(i)$ are generated in degree 0. Furthermore, let us recall that the *(right) standard module* $\Delta(i)$ is the largest quotient of the projective module $P(i)$ with no composition factors isomorphic to $S(j)$ for $j > i$. Hence each

$$\Delta(i) = e_i A / e_i A(e_{i+1} + e_{i+2} + \cdots + e_n)A$$

for $1 \leq i \leq n$ is a graded module generated in degree 0. Moreover, the (right) proper standard module $\overline{\Delta}(i)$ is the largest quotient of $P(i)$ with no composition factor of its radical isomorphic to $S(j)$ for $j \geq i$:

$$\overline{\Delta}(i) = e_i A / e_i (\operatorname{rad} A)(e_i + e_{i+1} + \cdots + e_n)A$$

is a graded module generated in degree 0. The corresponding left modules will be denoted by $\Delta^\circ(i)$ and $\overline{\Delta}^\circ(i)$.

The fact that an A-module X_A can be filtered by A-modules from a family $\mathcal{U}$ (i. e. that there is a chain $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$ of submodules of X_A so that $\bigcap_{i=0}^{\infty} X_i = 0$ and the respective factors X_i/X_{i+1} are of the form $\oplus_j U_j$ with $U_j \in \mathcal{U}$) will be denoted by $X_A \in \mathcal{F}(\mathcal{U})$.

Recall that $(A, \mathbf{e})$ is *standardly stratified* if $A_A \in \mathcal{F}(\Delta)$; it is *properly stratified* if $A_A \in \mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\Delta})$ (or, equivalently, $A_A \in \mathcal{F}(\Delta)$ and each $\Delta(i) \in \mathcal{F}(\overline{\Delta}(i))$; finally, it is *quasi-hereditary* if $A_A \in \mathcal{F}(\Delta)$ and $\Delta = \overline{\Delta}$. (Here Δ and $\overline{\Delta}$, respectively, denote the set of all standard and proper standard modules.)

Now, extending the idea of [D1] (see also [L]), we formulate a statement about standard and proper standard filtrations for an infinite dimensional basic K-algebra $(A, \mathbf{e})$. Since we will apply the result only for tightly graded K-algebras, we formulate the statement accordingly.

THEOREM 1. *Let $(A, \mathbf{e})$ be a tightly graded K-algebra. Then ${}_AA \in \mathcal{F}(\Delta^\circ)$ if and only if $A_A \in \mathcal{F}(\overline{\Delta})$. Moreover, in the case when $\dim_K(A) < \infty$, then writing $d_i = \dim_K(\mathrm{End}_A(S(i))$ we have*

$$\dim_K A = \sum_{i=1}^{n} \frac{1}{d_i} \cdot \dim_K \Delta^\circ(i) \cdot \dim_K \overline{\Delta}(i).$$

Proof. We can apply recursively the following two inequalities stated here for the graded trace ideal Ae_nA:

$$\begin{aligned}\dim_K((Ae_nA)_t) &\leq \sum_{u+s=t} \frac{1}{d_n} \dim_K((Ae_n)_u) \cdot \dim_K(\overline{\Delta}(n)_s) \\ &= \sum_{u+s=t} \frac{1}{d_n} \dim_K(\Delta^\circ(n)_u) \cdot \dim_K(\overline{\Delta}(n)_s),\end{aligned}$$

with equality if and only if $Ae_nA_A \in \mathcal{F}(\overline{\Delta}(n))$;

$$\begin{aligned}\dim_K((Ae_nA)_t) &\leq \sum_{u+s=t} \frac{1}{d_n} \dim_K((\mathrm{top}\, {}_AAe_nA)_s) \cdot \dim_K(\Delta^\circ(n)_u) \\ &= \sum_{u+s=t} \frac{1}{d_n} \dim_K((Ae_nA/(\mathrm{rad}\, A)e_nA)_s) \cdot \dim_K(\Delta^\circ(n)_u) \\ &= \sum_{u+s=t} \frac{1}{d_n} \dim_K(\overline{\Delta}(n)_s) \cdot \dim_K(\Delta^\circ(n)_u),\end{aligned}$$

with equality if and only if ${}_AAe_nA \in \mathcal{F}(\Delta^\circ(n))$. □

DEFINITION 2 (see [ADL3]). If $X = \bigoplus_{r \geq k} X_r$ is a graded right A-module over a (not necessarily tightly) graded algebra A, then the *Hilbert vector of* X is defined as the vector $H_A^X(q) = \left(H_1^X(q), \dots, H_n^X(q)\right) \in \mathbb{Z}[\![q, q^{-1}]\!]^n$ such that

$$H_i^X(q) = \sum_{r \geq k} [X_r : S(i)] \cdot q^r,$$

where $[X_r : S(i)]$ denotes the multiplicity of the simple module $S(i)$ in X_r, both considered as A_0-modules.

DEFINITION 3 (see [ADL3]). For a given ordered system $\mathbf{X} = (X(1), \dots, X(m))$ of graded A-modules $X(j)$ generated in degree 0, the *Hilbert matrix of* $\mathbf{X}$, denoted by $H_A^{\mathbf{X}}(q)$ is the matrix whose j-th column is the Hilbert vector $H_A^{X(j)}(q)$ of $X(j)$. In particular, $H_A(q) = H_A^{\mathbf{P}}(q)$ stands for the Hilbert matrix of the system $\mathbf{P} = (P(1), \dots, P(n))$, where $P(j)$ is the j-th indecomposable projective module.

Thus, $H_A^{\boldsymbol{\Delta}}(q)$, $H_A^{\overline{\boldsymbol{\Delta}}}(q)$, $H_{A^{opp}}^{\boldsymbol{\Delta}^\circ}(q)$ and $H_{A^{opp}}^{\overline{\boldsymbol{\Delta}}^\circ}(q)$ are the Hilbert matrices corresponding to the systems $\boldsymbol{\Delta} = (\Delta(1), \dots, \Delta(n))$, $\overline{\boldsymbol{\Delta}} = (\overline{\Delta}(1), \dots, \overline{\Delta}(n))$, $\boldsymbol{\Delta}^\circ = (\Delta^\circ(1), \dots, \Delta^\circ(n))$ and $\overline{\boldsymbol{\Delta}}^\circ = (\overline{\Delta}^\circ(1), \dots, \overline{\Delta}^\circ(n))$, respectively.

The following theorem gives a characterization of filtration properties of an algebra in terms of its Hilbert matrix.

THEOREM 4 (see [ADL3]). *Let $(A, \mathbf{e})$ be a tightly graded K-algebra. Then:*

(1) $A_A \in \mathcal{F}(\overline{\Delta})$, *that is $(A, \mathbf{e})$ is standardly stratified if and only if*

$$(\alpha) \qquad H_A(q) = H_A^{\overline{\mathbf{\Delta}}}(q) \cdot \left(D \cdot H_{A^{opp}}^{\mathbf{\Delta}^\circ}(q) \cdot D^{-1}\right)^T$$

where $D = D_A = (d_{ij})$ is the diagonal matrix with $d_{ii} = \dim_K End_A S(i)$.

(1′) $A_A \in \mathcal{F}(\Delta)$ *if and only if*

$$(\alpha'). \qquad H_A(q) = H_A^{\mathbf{\Delta}}(q) \cdot \left(D \cdot H_{A^{opp}}^{\overline{\mathbf{\Delta}}^\circ}(q) \cdot D^{-1}\right)^T$$

(2) $(A, \mathbf{e})$ *is properly stratified if and only if both (α) and (α') hold.*

(3) $(A, \mathbf{e})$ *is quasi-hereditary if and only if (α) holds and*

$$\det H_A(q) = 1.$$

Observe that if K is algebraically closed then $D = I_n$, the $n \times n$ identity matrix. Hence in this case the formula (α) takes the form $H_A(q) = H_A^{\overline{\mathbf{\Delta}}}(q) \cdot (H_{A^{opp}}^{\mathbf{\Delta}^\circ}(q))^T$. Let us also recall that

$$(\beta) \qquad H_{A^{opp}}(q) = D^{-1} \cdot (H_A(q))^T \cdot D$$

which in case that K is algebraically closed reduces to $H_{A^{opp}}(q) = (H_A(q))^T$.

Proof of Theorem 4. The statements (2) and (3) follow immediately from (1) and (1′). A sketch of a proof of (1) is available in [ADL3], Remark 1. For the benefit of the reader, let us provide the main steps of the proof that (α) holds if $(A, \mathbf{e})$ is properly stratified, in detail.

Let us denote by $\mathcal{X}(t)$ the n-tuple

$$(0, \dots, 0, \overline{\Delta}(t), 0, \dots, 0)$$

of A-modules with $\overline{\Delta}(t)$ in the t-th position. Similarly, write

$$\mathcal{Y}(t) = (0, \dots, 0, \Delta^\circ(t), 0, \dots, 0)$$

Note that $H_A^{\overline{\mathbf{\Delta}}}(q)$ is an upper triangular unimodular matrix and that

$$H_A^{\overline{\mathbf{\Delta}}}(q) = \sum_{t=1}^{n} H_A^{\mathcal{X}(t)}(q) = \sum_{t=1}^{n-1} H_B^{\overline{\mathcal{X}(t)}}(q) + H_A^{\mathcal{X}(n)}(q),$$

where $B = A/Ae_nA$ and $\overline{\mathcal{X}(t)}$ is the sequence of the first $n-1$ modules of $\mathcal{X}(t)$. Here and later on the first summand which is an $(n-1) \times (n-1)$ matrix is understood to be naturally embedded into an $n \times n$ matrix as its first $(n-1) \times (n-1)$ minor.

Similarly, the lower triangular matrix

$$D\cdot H^{\Delta^\circ}_{A^{opp}}(q)\cdot D^{-1}=\sum_{t=1}^{n} D\cdot H^{\mathcal{Y}(t)}_{A^{opp}}(q)\cdot D^{-1}=\sum_{t=1}^{n-1}\overline{D}\cdot H^{\overline{\mathcal{Y}}(t)}_{B^{opp}}(q)\cdot\overline{D}^{-1}+D\cdot H^{\mathcal{Y}(n)}_{A^{opp}}(q)\cdot D^{-1},$$

where $\overline{D}$ is the $(n-1)\times(n-1)$ diagonal matrix of the first $n-1$ rows (and columns) of D.

Thus, the right-hand side of (α) equals to

$$\sum_{t=1}^{n-1} H^{\overline{\mathcal{X}}(t)}_{B}\cdot\left(\overline{D}\cdot H^{\overline{\mathcal{Y}}(t)}_{B^{opp}}(q)\cdot\overline{D}^{-1}\right)^T+H^{\mathcal{X}(n)}_{A}(q)\cdot\left(D\cdot H^{\mathcal{Y}(n)}_{A^{opp}}(q)\cdot D^{-1}\right)^T.$$

Consequently, in order to prove (α), it is sufficient, by induction, to show that

$$H^{\mathcal{X}(n)}_{A}\cdot\left(D\cdot H^{\mathcal{Y}(n)}_{A^{opp}}(q)\cdot D^{-1}\right)^T=H^{Z}_{A}(q),$$

where $Z=(e_1Ae_nA,e_2Ae_nA,\dots,e_nAe_nA=e_nA)$. Hence, writing $\overline{\Delta}(n)=\oplus_{r\geq 0}\overline{\Delta}_r(n)$ and $\Delta^\circ(n)=\oplus_{s\geq 0}\Delta^\circ_s(n)$, we are supposed to show that, for each $1\leq i,j\leq n$,

$$\frac{d_{jj}}{d_{nn}}\cdot\sum_{r\geq 0}[\overline{\Delta}_r(n):S(i)]q^r\cdot\sum_{s\geq 0}[\Delta^\circ_s(n):S^\circ(j)]q^s=\sum_{t\geq 0}[P_t(j):S(i)]q^t.$$

But this equality is an immediate consequence of the following lemma.

Lemma 5. *Let $(e_1,e_2,\dots,e_n)$ be a complete sequence of orthogonal primitive idempotents of a graded standardly stratified algebra $A=\underset{r\geq 0}{\oplus}A_r$ Then, for each $1\leq i,j\leq n$,*

$$\frac{d_{jj}}{d_{nn}}\cdot\sum_{r=0}^{t}[\overline{\Delta}_{t-r}(n):S(i)]\cdot[\Delta^\circ_r(n):S^\circ(j)]=[(e_jAe_nA)_t:S(i)],$$

where $e_jAe_nA=\underset{t\geq 0}{\oplus}(e_jAe_nA)_t$.

Proof of Lemma 5. In view of the reciprocity law,

$$[\Delta^\circ(n):S^\circ(j)]=\frac{d_{nn}}{d_{jj}}\cdot[P(j):\overline{\Delta}(n)]=\frac{d_{nn}}{d_{jj}}\cdot[P(j):S(n)],$$

and thus

$$[\Delta^\circ_r(n):S^\circ(j)]=\frac{d_{nn}}{d_{jj}}\cdot[P_r(j):S(n)].$$

This is due to the fact that the factors $\overline{\Delta}(n)$ in a graded filtration of $P(j)$ by the proper standard modules are in one-to-one correspondence with the composition factors $S(n)$ of $P(j)$. Similarly, all composition factors $S(i)$ of e_jAe_nA are in one-to-one correspondence with the composition factors $S(i)$ of all $\overline{\Delta}(n)$ which appear in a filtration of e_jAe_nA. Moreover, the grading of such a composition factor is a sum of the grading of $S(n)$ and the grading of $S(i)$ in the respective copy of $\overline{\Delta}(n)$.

The proof of Lemma 5, and thus of (α), is completed. □

The Yoneda extension algebra A^* of the algebra A is the vector space $\bigoplus_{h\geq 0} \mathrm{Ext}_A^h(A_0, A_0)$ together with multiplication given by the Yoneda composition of extensions. Thus A^* has a natural grading given by the degree of Ext-modules. This grading is tight if and only if the algebra is Koszul. Let us recall that, by definition, a graded A-module X, generated in degree k, is *Koszul* if and only if X has a *linear projective resolution*, i.e. in the graded minimal projective resolution

$$\cdots \to P_h \to \cdots \to P_1 \to P_0 \to X \to 0,$$

the projective modules P_h are generated in degree $h+k$ for each $h \geq 0$. This is equivalent to the fact that all syzygies $\Omega_{h+1}(X)$ are top submodules of $\operatorname{rad} P_h(X)$. The algebra A is called *Koszul* if all (graded) simple modules are Koszul.

The natural contravariant functor $\mathrm{Ext}_A^* : \text{mod-}A \to A^*\text{-mod}$ is defined by

$$\mathrm{Ext}_A^*(X) = \bigoplus_{h\geq 0} \mathrm{Ext}^h(X, A_0) \quad \text{for every} \quad X \in \text{mod-}A,$$

where this decomposition provides also a grading of $\mathrm{Ext}_A^*(X)$. Note that the graded simple and indecomposable projective left A^*-modules can be obtained as $S^{*\circ}(i) = \mathrm{Ext}_A^*(P(i))$ and $P^{*\circ}(i) = \mathrm{Ext}_A^*(S(i))$, respectively. Let us consider in A^* the "opposite order" $\mathbf{f} = (f_n, f_{n-1}, \ldots, f_1)$ of idempotents, where $f(i) = \mathrm{id}_{S(i)}$, the identity map of $S(i)$. We shall denote the corresponding standard and proper standard modules with respect to this order by $\Delta^*(i)$, $\overline{\Delta}^*(i)$, $\Delta^{*\circ}(i)$ and $\overline{\Delta}^{*\circ}(i)$, respectively.

Definition 6. The *Poincaré vector* $P_A^X(q)$ of a graded A-module $X = \bigoplus_{r\geq 0} X_r$, generated in degree 0, is the vector $P_A^X(q) = \big(P_1^X(q), \ldots, P_n^X(q)\big)$, where

$$P_i^X(q) = \sum_{h\geq 0} (-1)^h \big[\, \mathrm{Ext}_A^h(X, A_0) : S^{*\circ}(i)\big]\, q^h.$$

One may define, as in the case of Hilbert vectors, the *Poincaré matrix* $P_A^{\mathbf{X}}(q)$ of a system $\mathbf{X}$ of modules.

Note that $P_A^X(q) \in \mathbb{Z}[\![q]\!]$ can be defined, equivalently, in terms of a minimal projective resolution of the A-module X, and that it is a polynomial vector if and only if the projective dimension of X is finite. Let us observe that for a Koszul algebra A we have

$$(\gamma) \qquad P_A^X(q) = H_{A^{*\,opp}}^{\mathrm{Ext}^*(X)}(-q).$$

In particular, $P_A^{\mathbf{S}}(q) = H_{A^{*\,opp}}(-q) = D^{-1} \cdot \big(H_{A^*}(-q)\big)^T \cdot D$. For simplicity, we shall denote this matrix by $P_A(q)$.

We can now formulate the following result.

Proposition 7 (see [ADL3]). *A graded A-module X, generated in degree 0, is Koszul if and only if*

$$H_A(q) \cdot P_A^X(q) = H_A^X(q).$$

In particular, a graded algebra A is Koszul if and only if

$$(\delta) \qquad H_A(q) \cdot P_A(q) = I_n.$$

Hence, for a Koszul algebra A, an A-module X_A is Koszul if and only if

$$(\varepsilon) \qquad P_A^X(q) = P_A(q) \cdot H_A^X(q).$$

Definition 8. A tightly graded standardly stratified Koszul algebra $(A, \mathbf{e})$ such that all its left standard as well as all its right proper standard modules are Koszul is called a *linearly stratified algebra.*

We should note that easy examples show that even in the quasi-hereditary case the Koszul property of standard modules on one side does not imply the Koszul property of standard modules on the other side (see, for example, 1.13 of [ADL4]).

To prepare the proof of the main theorem, let us show that the Ext^*-images of standard or proper standard modules are proper standard or standard, respectively.

Proposition 9. *Let $(A, \mathbf{e})$ be a tightly graded standardly stratified Koszul algebra.*

(1) *If all right proper standard A-modules are Koszul, then*

$$\mathrm{Ext}_A^*(\overline{\Delta}(i)) = \Delta^{*\circ}(i) \quad \textit{for all} \quad 1 \le i \le n.$$

(2) *If all left standard A-modules are Koszul, then*

$$\mathrm{Ext}_A^*(\Delta^\circ(i)) = \overline{\Delta}^*(i) \quad \textit{for all} \quad 1 \le i \le n.$$

Proof. (1) Denote by $M(i)$ the left A^*-module $\mathrm{Ext}_A^*(\Delta(i))$. Let us apply the Ext^* functor to the sequence $0 \to X \to P(i) \to \overline{\Delta}(i) \to 0$. Since $\overline{\Delta}(i)$ is Koszul, by Lemma 3.3 of [ADL1] we get an exact sequence of A^*-modules $0 \to \mathrm{Ext}_A^*(X) \to M(i) \to S{*}^\circ(i) \to 0$, where the embedding $\mathrm{Ext}_A^*(X) \to M(i)$ maps into the radical of $M(i)$. Thus $M(i)/\operatorname{rad} M(i) \cong S^{*\circ}(i)$. Moreover, by taking the long exact sequence of extensions on the short exact sequence $0 \to X \to P(i) \to \overline{\Delta}(i) \to 0$ and using the fact that $P(i) \in \mathcal{F}(\{\overline{\Delta}(i), \overline{\Delta}(i+1), \ldots, \overline{\Delta}(n)\})$, one gets that $\mathrm{Ext}_A^t(\overline{\Delta}(i), S(j)) = 0$ for all $j < i$ and $t \ge 0$. Hence all composition factors of $\mathrm{Ext}_A^*(\overline{\Delta}(i))$ are isomorphic to $S^{*\circ}(j)$ for some $j \ge i$. Thus there exists an epimorphism

$$\Delta^{*\circ}(i) \to M(i).$$

On the other hand, from the fact that if $\overline{\Delta}(i)$ is Koszul then $\operatorname{rad}\overline{\Delta}(i)$ is also Koszul (cf. for example [ADL4], Corollary 3.2), we get that the short exact sequence

$$0 \to \operatorname{rad}\overline{\Delta}(i) \to \overline{\Delta}(i) \to S(i) \to 0$$

yields, by Lemma 3.3 of [ADL1], a short exact sequence

$$0 \to \operatorname{Ext}^*_A(\operatorname{rad}\overline{\Delta}(i))[1] \to P^{*\circ}(i) \to M(i) \to 0.$$

Now, since $\operatorname{rad}\overline{\Delta}(i)$ is Koszul, any extension of $\operatorname{rad}\overline{\Delta}(i)$ by a simple module can be factored through the canonical epimorphism from $\operatorname{rad}\overline{\Delta}(i)$ to $\operatorname{rad}\overline{\Delta}(i)/\operatorname{rad}^2\overline{\Delta}(i)$, whose composition factors are isomorphic to simple modules $S(j)$ for $j < i$ (cf. [ADL1], Proposition 2.6). Consequently,

$$\operatorname{Ext}^*_A(\operatorname{rad}\overline{\Delta}(i)) \subseteq A^*(f_1 + \cdots + f_{i-1})A^* f_i,$$

and thus there is an epimorhism

$$M(i) \to \Delta^{*\circ}(i) = A^f_i / A^*(f_1 + \cdots + f_{i-1})A^* f_i.$$

The statement (1) follows.

(2) can be obtained by a similar argument. □

We are now ready to prove the main result of the paper.

THEOREM 10. *Let $(A, \mathbf{e})$ be a tightly graded algebra. Then $(A, \mathbf{e})$ is linearly stratified if and only if its Yoneda extension algebra $(A^*, \mathbf{f})$ is linearly stratified.*

Proof. In order to prove that the left standard modules and the right proper standard modules of A^* are Koszul, we follow similar arguments to those in [ADL4]. In particular, by Proposition 9

$$\operatorname{Ext}^*_A(\overline{\Delta}) = \Delta^{*\circ} \quad\text{and}\quad \operatorname{Ext}^*_A(\Delta^\circ) = \overline{\Delta}^*,$$

and these, as $\operatorname{Ext}^*_A(-)$-images of Koszul modules are Koszul. Consequently, we have to prove only that A^* is standardly stratified, i. e. $A^*_{A^*} \in \mathcal{F}(\overline{\Delta}^*)$. By (1) of Theorem 4 we are required to show that

$$(\alpha^*) \qquad H^{\overline{\Delta}^*}_{A^*}(q) \cdot \left(D \cdot H^{\Delta^{*\circ}}_{A^{*\,opp}}(q) \cdot D^{-1}\right)^T = H_{A^*}(q).$$

By (γ) and (ε) we have:

$$H^{\overline{\Delta}^*}_{A^*}(q) = H^{\operatorname{Ext}^*_{A^{opp}}(\Delta^\circ)}_{A^*}(q) = P^{\Delta^\circ}_{A^{opp}}(-q) = P_{A^{opp}}(-q) \cdot H^{\Delta^\circ}_{A^{opp}}(-q)$$

and

$$H^{\Delta^{*\circ}}_{A^{*\,opp}}(q) = H^{\operatorname{Ext}^*_A(\overline{\Delta})}_{A^{*\,opp}}(q) = P^{\overline{\Delta}}_A(-q) = P_A(-q) \cdot H^{\overline{\Delta}}_A(-q).$$

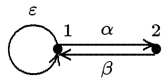
Thus, using (α), (β) and (δ), the left hand side of (α^*) equals, gradually, to

$$\begin{aligned}
&P_{A^{opp}}(-q) \cdot H^{\Delta^\circ}_{A^{opp}}(-q) \cdot D^{-1} \cdot H^{\overline{\Delta}}_A(-q)^T P_A(-q)^T \cdot D = \\
&= P_{A^{opp}}(-q) \cdot D^{-1} \cdot \left(D \cdot H^{\Delta^\circ}_{A^{opp}}(-q) \cdot D^{-1} \cdot H^{\overline{\Delta}}_A(-q)^T\right) P_A(-q)^T \cdot D = \\
&= P_{A^{opp}}(-q) \cdot D^{-1} \cdot H_A(-q)^T \cdot P_A(-q)^T \cdot D = \\
&= P_{A^{opp}}(-q) \cdot H_{A^{opp}}(-q) \cdot D^{-1} \cdot P_A(-q)^T \cdot D = \\
&= D^{-1} \cdot P_A(-q)^T \cdot D = \\
&= D^{-1} \cdot H_{A^{*\,opp}}(q)^T \cdot D = \\
&= H_{A^*}(q)
\end{aligned}$$

Hence $A^*_{A^*} \in \mathcal{F}(\overline{\Delta}^*)$, as required. □

Corollary 11. *A tightly graded algebra* $(A, \mathbf{e})$ *is properly stratified and all (right and left) standard and proper standard modules have linear projective resolutions if and only if the same conditions hold for its Yoneda extension algebra* $(A^*, \mathbf{f})$.

Example 12. The following simple example illustrates the necessity of conditions in Theorem 10 (and Corollary 11). Let us take the following quiver:



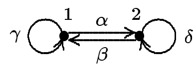
and consider $A = KQ/I$, where $I = \langle < \beta\alpha, \varepsilon\alpha, \varepsilon^2 \rangle >$. Then $(A, \mathbf{e})$ is a properly stratified Koszul algebra. However, $(A^*, \mathbf{f})$ is not standardly (and thus properly) stratified. Here, the left standard module $\Delta^\circ(1)$ does not have a linear projective resolution.

Let us conclude with two remarks.

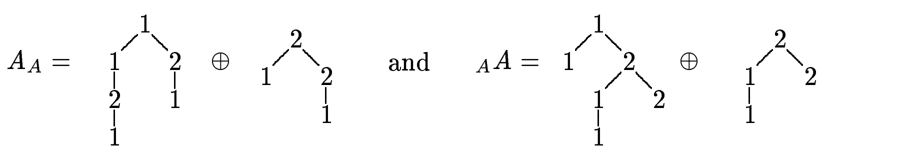
Remark 13. Let $(A, \mathbf{e})$ be a graded algebra, as before. Then the property that $(A, \mathbf{e})$ is standardly stratified algebra whose right proper standard modules have linear projective resolutions implies the smae property for each of the centralizers $C_i = (e_i + \cdots + e_n)A(e_i + \cdots + e_n)$, $1 \leq i \leq n$. This follows readily from the fact that a minimal projective resolution of $\overline{\Delta}(j)$ for $j \geq i$ is taken by the functor $\mathrm{Hom}_A((e_i + \cdots + e_n)A, -)$ to a minimal projective resolution of $\Delta(j)(e_i + \cdots + e_n)$ over C_i.

Example 14. Although the Hilbert and Poincaré matrices of the algebra [ADL3] reflect well Koszul properties of modules, they do not detect proper stratifications (i.e. without the appropriate Hilbert matrices of standard modules). Here are two Koszul algebras A_1 and A_2 whose Hilbert and Poincaré matrices are identical, although one is properly stratified and the other is not.

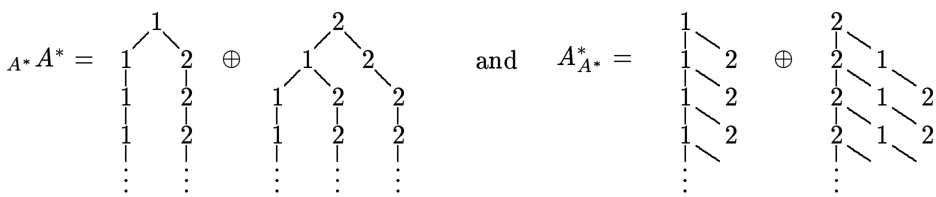
Let Q be the quiver



(1) Let $A = KQ/I$ where I is generated by $\{\gamma^2, \alpha\delta, \delta^2, \beta\alpha, \beta\gamma\}$. Then the respective regular representations are as follows:

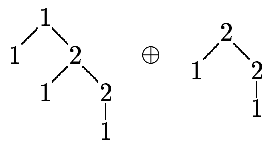


Hence we get the following extension algebra:

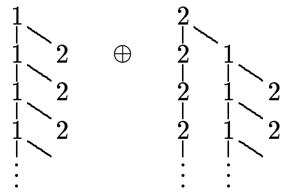


Thus both $(A, \mathbf{e})$, and $(A^*, \mathbf{f})$ are standardly stratified but not properly stratified.

(2) Let $B = KQ/I'$ where I is generated by $\{\gamma^2, \gamma\alpha, \delta^2, \beta\alpha, \beta\gamma\}$. The right regular decomposition of B_B is as follows:

$B_B =$ 

Here the left regular decomposition gives the same structure. For the extension algebra we get the following decomposition:

${}_{B^*}B^* =$ 

Thus $(B, \mathbf{e})$, as well as $(B^*, \mathbf{f})$ are standardly stratified. In fact, they are both properly stratified.

Here

$$H_A(q) = H_B(q) = \begin{bmatrix} 1+q+q^2+q^3 & q+q^2 \\ q+q^2 & 1+q \end{bmatrix} \quad \text{and}$$

$$P_A(q) = P_B(q) = \frac{1}{1+q} \cdot \begin{bmatrix} 1 & -q \\ -q & 1+q^2 \end{bmatrix}$$

Clearly, since both algebras are Koszul, we have $H_A(q)P_A(q) = I_n$.

References

[ADL1] Ágoston, I., Dlab, V., Lukács, E., Homological duality and quasi-heredity, *Can. J. Math.* **48** (1996), 897–917.

[ADL2] Ágoston, I., Dlab, V., Lukács, E., Stratified algebras, *C.R.Math.Rep. Acad.Sci.Canada* **20** (1998), 20–25.

[ADL3] Ágoston, I., Dlab, V., Lukács, E., Hilbert and Poincaré series of Koszul algebras, *C.R.Math.Rep. Acad.Sci.Canada* **23** (2001), 153–159.

[ADL4] Ágoston, I., Dlab, V., Lukács, E., Quasi-hereditary extension algebras, *Algebras and Represent. Theory* **6** (2003), 97–117.

[BGG] Bernstein, J., Gelfand, I.M, Gelfand, S.I., A category of $\mathfrak{g}$-modules, *Funct. Anal. Appl.* **10** (1976), 87–92.

[D1] Dlab, V., Quasi-hereditary algebras revisited, *An. St. Univ. Ovidius Constantza* **4** (1996), 43–54.

[D2] Dlab, V., Properly stratified algebras, *C.R. Acad. Sci. Paris Sér 1 Math.* **331** (2000), 191–196.

[FM] Frisk, A., Mazorchuk, V., Properly stratified algebras and tilting, *preprint (Uppsala University)* 2003, 1–35.

[FKM] Futorny, V., König, S., Mazorchuk, V., Categories of induced modules and standardly stratified algebras, *Algebras and Represent. Theory* **5** (2002), 259–276.

[KKM] Khomenko, O., König, S., Mazorchuk, V., Finitistic dimension and tilting modules for stratified algebras, *preprint (Uppsala University)* 2003, 1–21.

[L] Lakatos, P., On a theorem of Dlab, *Algebras and Repr. Theory* **3** (2000), 99–103.

[M] Mazorchuk, V., Stratified algebras arising in Lie theory, *preprint (Uppsala University)* 2002, 1–14.

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