Hilbert and Poincaré series of Koszul algebras

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Résumé. Cet article s'agit de quelques résultats récentes des auteurs en langage des séries de Hilbert et Poincaré d'une algébre graduée. En particulier, on montre qu'une algébre graduée Koszul est une algébre quasi-héréhitaire Koszul standarde si et seulement si son algébre (Yoneda) d'extension est une algébre quasi-héréditaire Koszul standarde.

Throughout the paper, $A = \underset{r\geq 0}{\oplus} A_r$ is a basic *positively graded* connected K-algebra, with $dim_K A_r < \infty$ for all r. Unless otherwise stated, we shall also assume that A is *tightly graded*. Thus A_0 is a finite dimensional semi-simple algebra and $A_r \cdot A_s = A_{r+s}$ for all integers $r, s \geq 0$. Obviously the (graded) radical of A is $\operatorname{rad} A = \underset{r\geq 1}{\oplus} A_r$. Let us fix a primitive orthogonal decomposition of the identity element $1 = e_1 + e_2 + \ldots + e_n$ so that $e_i \in A_0$ and keep the order $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ of this complete set of primitive orthogonal idempotents throughout the paper. By a *graded (right) A-module* X we shall always mean a vector space $X = \underset{r\geq k}{\oplus} X_r$ for some $k \in \mathbb{Z}$ with $X_r \cdot A_s \subseteq X_{r+s}$ where all X_r are finite dimensional. The module X is said to be *generated in degree k* if $X = X_k \cdot A$ (i.e. $X_k \cdot A_r = X_{k+r}$ for every $r \geq 0$). Note that in this case $X_r = 0$ for all $r < k$. A submodule Y of X is said to be a *graded submodule* of X if $Y = \underset{r\geq k}{\oplus} Y_r$ with $Y_r = Y \cap X_r$. If both X and Y are generated in the same degree k then Y is called a *top submodule* of X. If $X = \underset{r\geq k}{\oplus} X_r$ is a graded module generated in degree k, then the graded submodule $\operatorname{rad}^t X = \underset{r\geq t+k}{\oplus} X_r$ is generated in degree $t+k$ for all $t > 0$.

By a *graded morphism* $f : X \to Y$ between two graded A-modules we shall understand a module homomorphism of degree 0, given by a family of linear maps $f_r : X_r \to Y_r$ with obvious commuting properties. It is easy to see that, for a graded morphism $f : X \to Y$, the morphism $\operatorname{Ker} f \to X$ and $Y \to \operatorname{Coker} f$ are also graded, and if Y is generated in degree k, so is $\operatorname{Coker} f$. Every graded module X generated

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in degree k has a graded projective cover $P \to X \to 0$, where P is also generated in degree k.

Note that the right regular representation A_A and its indecomposable direct summands $P(i) = e_iA$ are graded modules generated in degree 0 and the graded submodules $\operatorname{rad}^t A_A$ and $\operatorname{rad}^t P(i)$ in degree $t \geq 1$. Similarly, the irreducible A-modules $S(i) = P(i)/\operatorname{rad} P(i)$ are generated in degree 0. Furthermore, let us recall that the *(right) standard module* $\Delta(i)$ is the largest quotient of the projective module $P(i)$ with no composition factors isomorphic to $S(j)$ for $j > i$. Hence

$$\Delta(i) = e_iA/e_iA(e_{i+1} + e_{i+2} + \cdots + e_n)A$$

for $1 \leq i \leq n$ is a graded module generated in degree 0.

Definition 1. If $X = \bigoplus_{r \geq k} X_r$ is a graded right A-module over a (not necessarily tightly) graded algebra A, then the *Hilbert vector of* X is defined as the vector $H_A^X(q) = \big(H_1^X(q), \ldots, H_n^X(q)\big) \in \mathbb{Z}[\![q, q^{-1}]\!]^n$ such that

$$H_i^X(q) = \sum_{r \geq k} [X_r : S(i)] \cdot q^r,$$

where $[X_r : S(i)]$ denotes the multiplicity of the simple module $S(i)$ in X_r, both considered as A_0-modules.

Definition 2. For a given ordered system $\mathbf{X} = (X(1), \ldots, X(m))$ of graded A-modules $X(j)$ generated in degree 0, the *Hilbert matrix of* $\mathbf{X}$, denoted by $H_A^{\mathbf{X}}(q)$ is the matrix whose j-th column is the Hilbert vector $H_A^{X(j)}(q)$ of $X(j)$. In particular, $H_A(q) = H_A^{\mathbf{P}}(q)$ stands for the Hilbert matrix of the system $\mathbf{P} = (P(1), \ldots, P(n))$, where $P(j)$ is the j-th indecomposable projective module. Similarly, $H_A^{\boldsymbol{\Delta}}(q)$ is the Hilbert matrix corresponding to the system $\boldsymbol{\Delta} = (\Delta(1), \ldots, \Delta(n))$ of the standard modules $\Delta(j)$.

Let us point out that the above defined vectors appear as columns in the forthcoming matrix calculations. Observe that $H_{A^{opp}}(q) = W_A^{-1} \cdot \big(H_A(q)\big)^T \cdot W_A$, where $W_A = (w_{ij})$ is the diagonal matrix with $w_{ii} = [\operatorname{End}_A S(i) : K]$; we shall use the symbol W_A to denote this matrix throughout the note. Let us also mention that $H_A^{\boldsymbol{\Delta}}(q)$ is an upper triangular matrix. Moreover, $H_A^{\mathbf{S}}(q) = I_n$, the $n \times n$ identity matrix, for the system of simple modules $\mathbf{S} = (S(1), \ldots, S(n))$. Finally, note that $H_A(1)$ is the Cartan matrix of the algebra A.

Recall that a finite dimensional algebra $(A, \mathbf{e})$ (i. e. the algebra A with respect to the given order $\mathbf{e}$ of the primitive idempotents e_i) is said to be a *quasi-hereditary algebra* if every indecomposable projective module $P(i)$ has a filtration whose factors are isomorphic to some $\Delta(j)$, $j \geq i$, and all standard modules $\Delta(i)$ are *Schurian* (i. e. the multiplicity $[\Delta(i) : S(i)] = 1$ for $1 \leq i \leq n$). Note that the latter condition is equivalent to the requirement that the diagonal elements of the Hilbert matrix $H_A^{\boldsymbol{\Delta}}(q)$ are all 1.

It is well-known that these properties imply the same properties for the corresponding (right) projective, standard and simple A-modules $P^\circ(i)$, $\Delta^\circ(i)$ and $S^\circ(i)$ of the opposite algebra A^{opp}.

PROPOSITION 1. *A finite dimensional algebra $(A, \mathbf{e})$ is quasi-hereditary if and only if*

$$H_A(q) = H_A^{\mathbf{\Delta}}(q) \cdot \left(W_A \cdot H_{A^{opp}}^{\mathbf{\Delta}^\circ}(q) \cdot W_A^{-1}\right)^T \quad and \quad \det H_A(q) = 1,$$

where $W_A = (w_{ij})$ is the diagonal matrix with $w_{ii} =$ $[\operatorname{End}_A S(i) : K]$.

Proof. The matrix equality ensures that the regular module has a filtration by standard modules, while the condition on the determinant of the Hilbert matrix gives that all standard modules are Schurian. This is a straightforward modification of the well-known numerical characterization of quasi-hereditary algebras (cf. [D], Theorem 2.4). □

REMARK 1. In fact, the above paper shows that the statement is true in a more general form: the algebra $(A, \mathbf{e})$ is *standardly stratified* (i. e. ${}_AA$ has a Δ°-filtration) if and only if

$$H_A(q) = H_A^{\overline{\mathbf{\Delta}}}(q) \cdot \left(W_A \cdot H_{A^{opp}}^{\mathbf{\Delta}^\circ}(q) \cdot W_A^{-1}\right)^T.$$

Here, $\overline{\mathbf{\Delta}} = \left(\overline{\Delta}(1), \dots, \overline{\Delta}(n)\right)$ is a sequence of the proper standard modules $\overline{\Delta}(i)$, defined as the largest quotient of $\Delta(i)$ with $[\overline{\Delta}(i) : S(i)] = 1$ (see also [ADL1]).

REMARK 2. Proposition 1 also shows that if $(A, \mathbf{e})$ is quasi-hereditary then its Hilbert matrix is the product of a lower and an upper unitriangular matrix. Simple examples show that this property is far from characterizing quasi-heredity of algebras.

Recall that the Yoneda extension algebra A^* of the algebra A is the vector space $\bigoplus_{h \geq 0} \operatorname{Ext}_A^h(\hat{S}, \hat{S})$ together with multiplication given by the Yoneda composition of extensions; here, $\hat{S} = \bigoplus_{i=1}^{n} S(i)$. Thus A^* has a natural grading given by the degree of Ext-modules. This grading is tight if and only if the algebra is Koszul. Let us recall that, by definition, a graded A-module X, generated in degree k is *Koszul* if and only if X has a *linear projective resolution*, i. e. in the graded minimal projective resolution

$$\cdots \to P_h \to \cdots \to P_1 \to P_0 \to X \to 0$$

the projective modules P_h are generated in degree $h + k$ for each $h \geq 0$. This is equivalent to the fact that all syzygies $\Omega_{h+1}(X)$ are top submodules of $\operatorname{rad} P_h(X)$. The finite dimensional algebra A is called *Koszul* if all simple modules are Koszul. Thus A is Koszul if and only if, for all $1 \leq i, j \leq n$,

$$\operatorname{Ext}_A^h(S(i)[s], S(j)[r]) \neq 0 \text{ implies } r - s = h.$$

Here, $S(k)[t]$ denotes the graded module $S(k)$, shifted by t.

The natural contravariant functor $\operatorname{Ext}_A^* : \text{mod-}A \to A^*\text{-mod}$ is defined by

$$\operatorname{Ext}_A^*(X) = \bigoplus_{h \geq 0} \operatorname{Ext}^h(X, \hat{S}) \quad \text{for every} \quad X \in \text{mod-}A,$$

where this decomposition provides also a grading of $\operatorname{Ext}_A^*(X)$. Note that the graded simple and indecomposable projective left A^*-modules can be obtained as $S^{*\circ}(i) = \operatorname{Ext}_A^*(P(i))$ and $P^{*\circ}(i) = \operatorname{Ext}_A^*(S(i))$, respectively.

DEFINITION 3. The *Poincaré vector* $P_A^X(q)$ of a graded A-module $X = \underset{r\geq 0}{\oplus} X_r$, generated in degree 0 is the vector $P_A^X(q) = \big(P_1^X(q), \dots, P_n^X(q)\big)$, where

$$P_i^X(q) = \sum_{h\geq 0} (-1)^h \big[\, \mathrm{Ext}_A^h(X, \hat{S}) : S^{*\circ}(i)\big] q^h.$$

One may define, as in the case of Hilbert vectors, the *Poincaré matrix* $P_A^{\mathbf{X}}(q)$ of a system $\mathbf{X}$ of modules.

Note that $P_A^X(q) \in \mathbb{Z}[\![q]\!]$ can be defined, equivalently, in terms of a minimal projective resolution of the A-module X, and that it is a polynomial vector if and only if the projective dimension of X is finite. Let us observe that $P_A^X(q) = H_{A^{*\,opp}}^{\mathrm{Ext}^*(X)}(-q)$. In particular, $P_A^{\mathbf{S}}(q) = H_{A^{*\,opp}}(-q) = W_A^{-1} \cdot \big(H_{A^*}(-q)\big)^T \cdot W_A$. For simplicity, we shall denote this matrix by $P_A(q)$. Let us note that, again, $P_{A^{opp}}(q) = W_A^{-1} \cdot \big(P_A(q)\big)^T \cdot W_A$.

We can now formulate the following result.

PROPOSITION 2. ([BG]) *The graded A-module X, generated in degree 0 is Koszul if and only if*

$$H_A(q) \cdot P_A^X(q) = H_A^X(q).$$

Proof. Consider a minimal graded projective resolution

$$\cdots \to P_h \to \cdots \to P_1 \to P_0 \to X \to 0$$

of the module X. Define the power series vector (formally) by

$$R_A^X(q) = \big(R_1^X(q), \dots, R_n^X(q)\big) = \sum_{h\geq 0} (-1)^h \cdot H_A^{\mathrm{top}\, P_h}(q);$$

here $\mathrm{top}\, P_h$ is the semisimple graded module $P_h / \operatorname{rad} P_h$. Note that the $r-$th component $(P_h)_r$ of P_h is 0 for all $r < h$. Consequently, each component of $H_A^{\mathrm{top}\, P_h}(q)$ is divisible by q^h. Using Euler's argument we get easily that

$$H_A(q) \cdot R_A^X(q) = H_A^{P_0}(q) - H_A^{P_1}(q) + \cdots + (-1)^h H_A^{P_h}(q) + \cdots = H_A^X(q).$$

On the other hand, it is clear that $P_i^X(q) = \sum_{h\geq 0} [\mathrm{top}\, P_h : S_i] \cdot (-q)^h$. So one can see easily that X is Koszul if and only if $R_A^X(q) = P_A^X(q)$. Finally, the statement follows by observing that $H_A(q)$ is an invertible matrix, since $H_A(0) = I_n$. $\square$

COROLLARY 1. *A graded algebra A is Koszul if and only if*

$$H_A(q) \cdot P_A(q) = I_n.$$

Note that $H_A(q)\cdot R_A^{\mathbf{S}}(q) = H_A^{\mathbf{S}}(q) = I_n$. Thus, if $gl.dim\, A < \infty$, then $\det R_A^{\mathbf{S}}(q)$ and $\det H_A(q)$ belong to $\mathbb{Z}[q]$, and therefore $\det H_A(q) = \pm 1$. On the other hand, $\det H_A(0) = 1$; hence we obtain the following well-known result.

PROPOSITION 3. ([W]) *If the graded algebra A has finite global dimension, then $\det H_A(q) = 1$, consequently the Cartan determinant of A is 1.* □

Recall that an algebra $(A, \mathbf{e})$ is said to be *standard Koszul* if all right and left standard modules are Koszul. It is one of the main results of [ADL2] that a standard Koszul quasi-hereditary algebra is Koszul. We can reformulate this result in terms of Hilbert and Poincaré matrices in the following way.

THEOREM 1. ([ADL2]) *Let us assume that for the algebra $(A, \mathbf{e})$ the following conditions hold:*
(i) $H_A(q) = H_A^{\mathbf{\Delta}}(q) \cdot \left(W_A \cdot H_{A^{opp}}^{\mathbf{\Delta}^\circ}(q) \cdot W_A^{-1}\right)^T$, *and* $\det H_A(q) = 1$.
(ii) $H_A(q) \cdot P_A^{\mathbf{\Delta}}(q) = H_A^{\mathbf{\Delta}}(q)$ *and* $H_{A^{opp}}(q) \cdot P_{A^{opp}}^{\mathbf{\Delta}^\circ}(q) = H_{A^{opp}}^{\mathbf{\Delta}^\circ}(q)$.
Then, $H_A(q) \cdot P_A(q) = I_n$.

A simple matrix calculation leads to the following statement.

COROLLARY 2. *Let $(A, \mathbf{e})$ be a graded algebra satisfying the conditions (i) and (ii) of Theorem 1. Then the condition (i) holds also for the Poincaré matrices:*

$$P_A(q) = P_A^{\mathbf{\Delta}}(q) \cdot \left(W_A \cdot P_{A^{opp}}^{\mathbf{\Delta}^\circ}(q) \cdot W_A^{-1}\right)^T \textit{and } \det P_A(q) = 1.$$

Proof. Substituting the expressions in (ii) into (i), we get

$$H_A(q) = \left(H_A(q) \cdot P_A^{\mathbf{\Delta}}(q)\right) \cdot \left(W_A \cdot H_{A^{opp}}(q) \cdot P_{A^{opp}}^{\mathbf{\Delta}^\circ}(q) \cdot W_A^{-1}\right)^T.$$

Rewriting this equality,

$$I_n = P_A^{\mathbf{\Delta}}(q) \cdot W_A^{-1} \cdot \left(P_{A^{opp}}^{\mathbf{\Delta}^\circ}(q)\right)^T \cdot W_A \cdot W_A^{-1} \cdot \left(H_{A^{opp}}(q)\right)^T \cdot W_A,$$

and thus

$$I_n = P_A^{\mathbf{\Delta}}(q) \cdot \left(W_A \cdot P_{A^{opp}}^{\mathbf{\Delta}^\circ}(q) \cdot W_A^{-1}\right)^T \cdot H_A(q).$$

By Theorem 1, the inverse of the matrix $H_A(q)$ is $P_A(q)$. This yields the desired formula. Clearly $\det P_A(q) = \det\left(H_A(q)\right)^{-1} = 1$. □

Now we can give an alternative proof of one of the main statements of [ADL2].

THEOREM 2. *Let $(A, \mathbf{e})$ be a graded standard Koszul quasi-hereditary algebra. Then the extension algebra $(A^*, \mathbf{f})$ (with respect to the opposite order of the idempotents $\mathbf{f} = (f_n, f_{n-1}, \dots, f_1)$, where $f_i = \mathrm{id}_{S(i)}$) is also a standard Koszul quasi-hereditary algebra.*

Proof. In order to identify the standard modules over A^*, we need the following characterization of standard modules in terms of Hilbert and Poincaré matrices.

PROPOSITION 4. *A system* $\mathbf{X} = (X(1), \dots, X(n))$ *of* A*-modules generated in degree* 0 *is isomorphic to the system of the standard modules* $\mathbf{\Delta} = (\Delta(1), \dots, \Delta(n))$ *of* $(A, \mathbf{e})$ *if and only if the matrix* $H_A^{\mathbf{X}}(q)$ *is an upper triangular matrix with* $H_A^{\mathbf{X}}(0) = I_n$*, while in the matrix* $P_A^{\mathbf{X}}(q) - I_n$ *all elements above and on the main diagonal are divisible by* q^2*. In particular, if* $(A, \mathbf{e})$ *is quasi-hereditary, then the matrices* $H_A^{\mathbf{\Delta}}(q)$ *and* $H_{A^{opp}}^{\mathbf{\Delta}^\circ}(q)$ *are upper unitriangular and the matrices* $P_A^{\mathbf{\Delta}}(q)$ *and* $P_{A^{opp}}^{\mathbf{\Delta}^\circ}(q)$ *are lower unitriangular.*

Proof. It is clear that the system of standard modules satisfies the given conditions. For the converse, the fact that the Hilbert matrix of $\mathbf{X}$ is upper triangular means that $X(i)$ has no composition factors of index larger than i while the condition on the constant part of the matrix ensures that each module $X(i)$ is local. Finally, the last condition yields $\mathrm{Ext}_A^1(X(i), S(j)) = 0$ for $j \leq i$, implying that $X(i)$ is a maximal quotient of $\Delta(i)$, i. e. $X(i) = \Delta(i)$. The last statement follows from well-known properties of standard modules over quasi-hereditary algebras. □

COROLLARY 3. *If* $(A, \mathbf{e})$ *is a graded standard Koszul quasi-hereditary algebra, then* $\mathrm{Ext}_A^*\left(\Delta(i)\right) = \Delta^{*\circ}(i)$ *and* $\mathrm{Ext}_A^*\left(\Delta^\circ(i)\right) = \Delta^*(i)$ *are the left and right standard modules over* $(A^*, \mathbf{f})$*, respectively.*

Proof. We make use of the following facts. The left A^*-module $\mathrm{Ext}^*(X)$ of a right graded Koszul module X over a Koszul algebra A, is Koszul, and furthemore, $X \cong \mathrm{Ext}^*(\mathrm{Ext}^*(X))$ if we identify A with the isomorphic algebra A^{**}.

Thus, while $P_A^{\mathbf{\Delta}}(q) = H_{A^{*opp}}^{\mathrm{Ext}^*(\mathbf{\Delta})}(-q)$ is true in general, the Koszul property of the standard modules now implies that $P_{A^{*opp}}^{\mathrm{Ext}^*(\mathbf{\Delta})}(q) = H_{A^{**}}^{\mathrm{Ext}^*(\mathrm{Ext}^*(\mathbf{\Delta}))}(-q) = H_A^{\mathbf{\Delta}}(-q)$, and the same relations hold for the left standard modules. Thus by Proposition 4, we get that $\mathrm{Ext}^*(\Delta^\circ(1)), \dots, \mathrm{Ext}^*(\Delta^\circ(n))$ are the right, while $\mathrm{Ext}^*(\Delta(1)), \dots, \mathrm{Ext}^*(\Delta(n))$ are the left standard modules over $(A^*, \mathbf{f})$. □

In order to finish the proof of Theorem 2, we need only to note that, by our earlier observation, Corollary 3 implies that the left and right standard modules over $(A^*, \mathbf{f})$ are Koszul. Furthermore, by Corollary 2 and Proposition 1, we get that $(A^*, \mathbf{f})$ is quasi-hereditary. Thus A^* is a standard Koszul quasi-hereditary algebra, as required. □

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