# REALIZATIONS OF FROBENIUS FUNCTIONS

István Ágoston[1], Erzsébet Lukács[2] and Claus Michael Ringel

Abstract. In an earlier paper [ALR] we studied the so-called Frobenius functions on certain translation quivers. Here we show that the classification given there is in some sense complete: every Frobenius length function on the wing $W(n)$ and the tube $T(n)$ is equivalent to the length function on a convex subquiver of the Auslander–Reiten quiver of the module category over some algebra $A$.

## 1. Introduction

Let $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ be a translation quiver (without multiple arrows), and let $f : \Gamma_0 \longrightarrow \mathbb{Z}$ be an integral valued function defined on the vertices of $\Gamma$. For any $z \in \Gamma_0$ a non-projective vertex we define the defect of the function $f$ at the vertex $z$ (or rather on the mesh ending at $z$) by $\delta(z) = \delta_f(z) = f(z) + f(\tau z) - \sum_{y \to z} f(y)$. We call the function $f$ a *Frobenius function* if $\delta(z) \neq 0$ implies $\delta(z) > f(z)$ and $\delta(z) > f(\tau z)$. A Frobenius function with positive values only will said to be *positive*. Meshes with non-zero defect are called *incomplete meshes*; otherwise we say that $f$ is additive on the mesh. Two Frobenius functions are said to be *equivalent* provided they have the same set of incomplete meshes. A typical example of a positive Frobenius function is the dimension function on the stable Auslander–Reiten quiver of a finite dimensional selfinjective algebra over a field $k$. A *Frobenius length function* is a positive Frobenius function for which $\delta(z) - f(z) = \delta(z) - f(\tau z) = 1$ whenever the mesh ending at the vertex $z$ is incomplete. A typical example of a Frobenius (length) function is the length function $\ell$ on the stable Auslander–Reiten quiver of a finite dimensional selfinjective algebra over a field $k$.

In [ALR] we studied in detail positive Frobenius functions on certain translation quivers. In particular, we investigated the cases of the wings $W(n)$, the stable tubes $T(n)$ and the related translation quivers $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty^\infty$ and gave a full description of the equivalence classes of Frobenius functions on these quivers by describing geometrically the possible configurations of incomplete meshes. The aim of this paper is to give algebraic realizations for Frobenius length functions on $W(n)$ and $T(n)$. We want to prove the following two theorems.

---

1991 Mathematics Subject Classification. Primary 16G70, 16G20.

[1] Research partially supported by the Foundation for Hungarian Higher Education and Research and by Hungarian NFSR grant no. T023434

[2] Research partially supported by Hungarian NFSR grant no. T016432

THEOREM 1. *Let $f$ be a positive Frobenius length function on the translation quiver $W(n)$. Then there exists a finite dimensional, basic special biserial algebra $\mathcal{C}$ and a translation subquiver $\Gamma$ of the Auslander–Reiten quiver $\Gamma(\mathcal{C})$, isomorphic to (hence may be identified with) $W(n)$, so that:*

(*i*) *the length function on $\Gamma$ is equivalent to $f$;*
(*ii*) *the incomplete meshes of $\Gamma$ with respect to $f$ are precisely those from which a projective-injective vertex in the Auslander–Reiten quiver $\Gamma(\mathcal{C})$ has been removed to obtain $\Gamma$.*

THEOREM 2. *Let $f$ be a positive Frobenius length function on the translation quiver $T(n)$. Then there exists a finite dimensional special biserial algebra $\mathcal{C}$ and a translation subquiver $\Gamma$ of the Auslander–Reiten quiver $\Gamma(\mathcal{C})$, isomorphic to (hence may be identified with) $T(n)$, so that:*

(*i*) *the length function on $\Gamma$ is equivalent to $f$;*
(*ii*) *$\Gamma$ can be obtained from a component of $\Gamma(\mathcal{C})$ by removing all the projective-injective vertices;*
(*iii*) *the incomplete meshes of $\Gamma$ with respect to $f$ are precisely those from which a projective-injective vertex in the Auslander–Reiten quiver $\Gamma(\mathcal{C})$ has been removed.*

Both theorems assert that we may realize positive Frobenius length functions at least up to equivalence. It is easy to see that for a stable tube $T(n)$ not all positive Frobenius length functions themselves can be realized in this way. For a wing $W(n)$, our proof of Theorem 1 will give a realization of all positive Frobenius length functions which are basic in the sense of [ALR] (we are going to recall the definition below). Note that every Frobenius length function on $W(n)$ is equivalent to a basic Frobenius length function [ALR].

Let us fix some of the notation used throughout the paper. In particular, we will need a coordinatization for the relevant translation quivers: the wings $W(n)$ and the stable tubes $T(n)$, but also $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty^\infty$. The set of vertices of $\mathbb{Z}A_\infty^\infty$ is the set $\mathbb{Z}\times\mathbb{Z}$ of integral lattice points in the plane. There are arrows $(i,j)\to(i+1,j)$ and $(i,j)\to(i,j+1)$, and the translation is defined by $\tau(i,j)=(i-1,j-1)$ for any $(i,j)\in\mathbb{Z}\times\mathbb{Z}$. We introduce $\mathbb{Z}A_\infty$ as the full subquiver of $\mathbb{Z}A_\infty^\infty$ on the set of vertices $\{(i,j)\in\mathbb{Z}\times\mathbb{Z}\mid i\le j\}$. For $W(n)$ we fix a standard embedding, mapping the projective vertices $\mathbf{p}_1,\mathbf{p}_2,\dots,\mathbf{p}_n$ to the points $(1,1),(1,2),\dots,(1,n)$. Hence, $W(n)=\{(i,j)\mid 1\le i\le j\le n\}$. Finally, we also consider $\mathbb{Z}A_\infty$ as the universal cover of $T(n)$ so that the vertices on the mouth of $T(n)$ correspond to the boundary $\{(i,i)\mid i\in\mathbb{Z}\}$ of $\mathbb{Z}A_\infty$. The standard coordinatization of $T(n)$ maps the points of $T(n)$ to the strip $\{(i,j)\mid 1\le i\le j,\ i\le n\}$.

Recall from [ALR] that to any Frobenius function $f$ on the wing $W(n)$ we can attach a *binary vector* $(f_1,f_2,\dots,f_n)=\big(f(\mathbf{p}_1),f(\mathbf{p}_2),\dots,f(\mathbf{p}_n)\big)$, (i.e. a vector where $f_i=f_j$ for $i<j$ implies that there is an index $\ell$ such that $i<\ell<j$ and $f_\ell<f_i$) and to this a *rooted embedded binary tree* $B(f)$ on the vertex set $\{1,2,\dots,n\}$ by the following recursion. If $f_r$ is the (unique) minimal element of the binary vector, then the root of the tree will be the vertex $r$. Furthermore, we put an arrow going from $r$ to the root of the binary tree assigned to the vector $(f_1-f_r,\dots,f_{r-1}-f_r)$ and an arrow going from the root of the binary tree assigned to $(f_{r+1}-f_r,\dots,f_n-f_r)$ into the vertex $r$. The first arrow (if it exists) will be

'coloured' by $\varphi$, and the second one (if it exists) by $\psi$. The arrows coloured by $\varphi$ will be called $\varphi$-arrows, those coloured by $\psi$ are the $\psi$-arrows. A binary vector is called *basic* if the minimal elements of the binary vectors considered in the above recursion are always 1. It is easy to see that if $(f_1, f_2, \ldots, f_n)$ is a basic binary vector, and $B(f)$ the associated binary tree, then $f_i$ is equal to the number of vertices in the unique path in the underlying unoriented graph of $B(f)$ from the vertex $i$ to the root $r$. (Note that the rooted embedded binary tree $B(f)$ was denoted by $\vec{B}(f)$ in [ALR], but here we will use this simplified notation.) A positive Frobenius function on $W(n)$ is called *basic* if the restriction of $f$ to the projective vertices gives a basic binary vector.

The structure of the paper is as follows: we describe the constructions in Section 2, first those concerning the realization of Frobenius functions on a wing $W(n)$, then those dealing with the tube $T(n)$. All proofs are deferred to Section 3. We would like to mention that the ideas used in the paper rely very much on ideas contained in the works [HW] and [WW]. In particular we use the explicit description of the Auslander–Reiten sequences for special biserial algebras, given in [WW] (see also [BR]). For unexplained notation and background we refer to [ALR] and [R].

## 2. Construction of the algebras

In what follows, $k$ will be an arbitrary field. We will consider $k$-algebras (not necessarily finite dimensional) which will be given as path algebras of some finite quiver modulo some relations.

Thus let $\mathcal{C}$ be such a $k$-algebra. The $\mathcal{C}$-modules considered will always be left modules. We will 'multiply the arrows' of the quiver from the left: if $\alpha$ and $\beta$ are arrows in the quiver, then $\alpha\beta$ will stand for the path with $\beta$ followed by $\alpha$. The category of all left $\mathcal{C}$-modules which are finite dimensional over $k$ will be denoted by $\mathcal{C}$-mod.

The aim of this section is to describe in detail some finite dimensional, basic special biserial algebras $\mathcal{C}$ and part of their Auslander–Reiten quivers. These are the algebras whose existence is asserted in Theorem 1 and Theorem 2. The constructions presented here are complete, but proofs are deferred to Section 3. We separate the two cases, first we will deal with a wing $W(n)$, then the construction will be modified in order to cover the case of a stable tube $T(n)$. In addition, a typical example is exhibited in detail in Section 2 and used as an illustration throughout the paper.

**The case of a wing $\mathbf{W(n)}$.** We fix a basic Frobenius length function on $W(n)$ and are going to exhibit an algebra $\mathcal{C}$ satisfying the requirements of Theorem 1. First, we will construct the quiver of $\mathcal{C}$.

We start with the binary tree $B = B(f)$ corresponding to the basic binary vector $(f_1, f_2, \ldots, f_n)$ as defined above and let $\mathcal{B}$ be the algebra given by the quiver $B = B(f)$ together with the relations $\varphi\psi = 0$ for all arrows of type $\varphi$ and $\psi$. We denote by $r$ the root of $B$. (Note that it is well known how to attach to $f$ a basic tilting $A_n(k)$-module with dimension vector $(f_1, f_2, \ldots, f_n)$; here $A_n(k)$ is the quiver algebra over $k$ for the linearly ordered quiver of type $A_n$; see [HR]. The algebra $\mathcal{B}$ is just the corresponding tilted algebra.)

We define the convex subquiver $A^+$ of $B$ formed by the endpoints of the $\varphi$-arrows in $B$, and let $A^0$ be the set of vertices of $B$ not belonging to $A^+$. Notice that

$A^+$ is the disjoint union of the maximal $\varphi$-paths of $B$ without their starting points. In order to stress that a vertex $i$ belongs to $A^+$, we will sometimes write $i^+$ instead of $i$. Let $A^-$ be an additional copy of the quiver $A^+$, hence isomorphic to $A^+$, with the correspondence $i \mapsto i^-$ giving the isomorphism (the inverse isomorphism from $A^-$ to $A^+$ being denoted by $j \mapsto j^+$).

Finally we define the quiver $C$ on the disjoint union of $B$ and $A^-$ with additional arrows going from $A^-$ to $B$ as follows. If there is a maximal $\varphi$-path in $B$ of length at least 1 starting in $i$ and ending in $j$, then $j$ belongs to $A^+$ and we add a $\varphi$-arrow from $j^-$ to $i$. If there is a maximal $\psi$-path starting in $i$ and ending in $j$, and if $j$ belongs to $A^+$, then we add an $\psi$-arrow from $j^-$ to $i$.

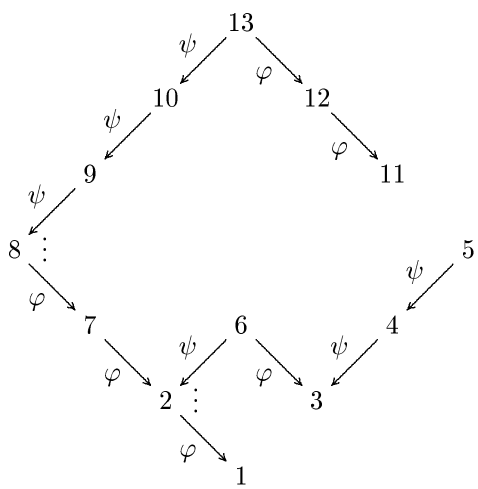
Thus for the sets of vertices we have $C = A^+ \cup A^0 \cup A^-$, and $B$ is the full subquiver of $C$ generated by $A^+ \cup A^0$. We will denote by $B^-$ the full subquiver of $C$ generated by $A^0 \cup A^-$. To emphasize a duality between $B$ and $B^-$, we occassionally write $B = B^+$.

We note the following property of $C$. Every vertex is the starting point of at most one $\varphi$-arrow and at most one $\psi$-arrow, and it is also the end point of at most one $\varphi$-arrow and at most one $\psi$-arrow.

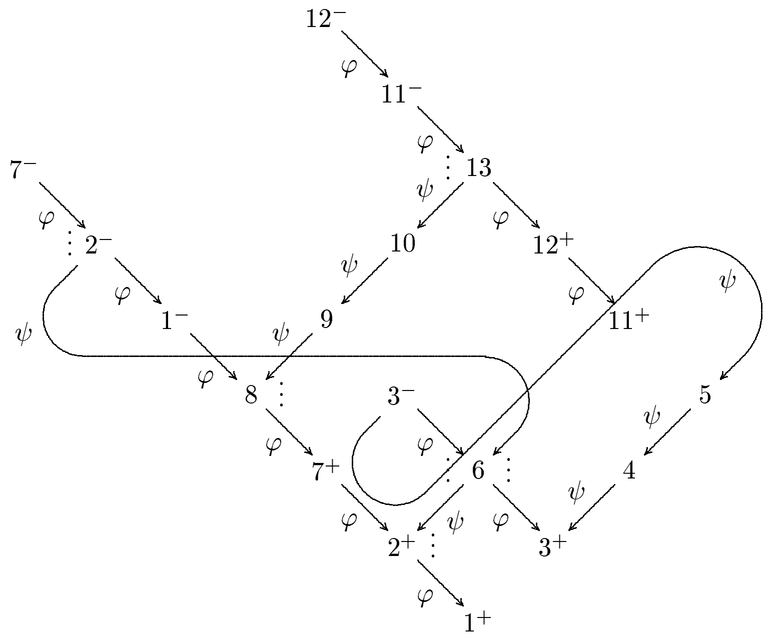
Example. Start with the basic binary vector

$$(4, 3, 5, 6, 7, 4, 2, 1, 2, 3, 6, 5, 4).$$

The quiver $B(f)$ has the following shape (the dots indicate where a $\varphi$-arrow follows a $\psi$-arrow). In contrast to a usual convention, the root of the tree (vertex 8) is drawn at the left, not at the bottom of the picture; in this way, all arrows point downwards (this will be helpful for visualizing 'strings').




The subquiver $A^+$ has three components, namely $7 \to 2 \to 1$, $3$ and $12 \to 11$, therefore the quiver of $C$ is as follows (again we have added dots in order to indicate the positions where a $\varphi$-arrow and a $\psi$-arrow follow each other, in any order). Some arrows are drawn curved in order to indicate the local similarity.




We define the algebra $\mathcal{C}$ to be the path-algebra of the quiver $C$ modulo the relations $\varphi\psi = 0$ and $\psi\varphi = 0$ for all arrows of type $\varphi$ and $\psi$ in $C$, all commutativity relations for the paths going from $i^-$ to $i^+$ for $i \in A^+$, plus the following zero relations: if $i, j \in A^+$ with an arrow $i \xrightarrow{\varphi} j$ and $\alpha$ is the $\varphi$-path from $i^-$ to $i^+$, then we set the product $\varphi\alpha = 0$.

We note here that, alternatively, we could have defined $\mathcal{C}$ as the matrix algebra

$$\mathcal{C} = \begin{bmatrix} \mathcal{B} & I^+ \\ 0 & \mathcal{A}^- \end{bmatrix},$$

where $\mathcal{A}^-$ is the path algebra of $A^-$, the bimodule $I^+$ is the direct sum of the indecomposable injective $\mathcal{B}$-modules corresponding to the vertices in $A^+$, with the right action of $\mathcal{A}^-$ on $I^+$ given by the canonical isomorphism $\mathcal{A}^- \simeq \operatorname{End}_{\mathcal{B}}(I^+)$.

We have completed the construction of $\mathcal{C}$. In order to verify that $\mathcal{C}$ is as required, we need to consider its Auslander–Reiten quiver. Given a vertex $i$, we denote by $P_{\mathcal{C}}(i)$ the indecomposable projective $\mathcal{C}$-module corresponding to $i$ and by $I_{\mathcal{C}}(i)$ the indecomposable injective $\mathcal{C}$-module corresponding to $i$ (similar notation will be used when we consider indecomposable projective or injective modules over other algebras). It is easy to see that $I_{\mathcal{C}}(i) = P_{\mathcal{C}}(i^-)$ is a projective-injective $\mathcal{C}$-module for every $i \in A^+$. We claim that the convex subquiver of the Auslander–Reiten quiver of $\mathcal{C}$ spanned by $P_{\mathcal{C}}(r)$ and $I_{\mathcal{C}}(n)$ with the projective-injective modules $I_{\mathcal{C}}(i)$ for $i \in A^+$ removed, gives a required representation of the Frobenius function $f$.

In order to verify this claim, we will work with the following factor algebra $\overline{\mathcal{C}} = \mathcal{C}/J$ of $\mathcal{C}$. Let $J$ be the direct sum of the socles of the projective $\mathcal{C}$-modules $P_{\mathcal{C}}(i^-)$ with $i \in A^+$. This is an ideal of $\mathcal{C}$. Since the modules $P_{\mathcal{C}}(i^-)$ with $i \in A^+$ are projective-injective, the ideal $J$ annihilates every other indecomposable module. The quiver of $\overline{\mathcal{C}}$ remains the same as that of $\mathcal{C}$, i. e. it is $C$, and besides the relations for $\mathcal{C}$, we add the new relations $\alpha = 0$ for every path $\alpha$ going from $i^-$ to $i$ for some

$i \in A^+$. These paths will be called *Nakayama paths*. The only difference between the Auslander–Reiten quivers of $\overline{\mathcal{C}}$ and $\mathcal{C}$ is that in the latter we have removed the projective-injective modules $I_{\mathcal{C}}(i) = P_{\mathcal{C}}(i^-)$ with $i \in A^+$, and have broken the $\tau$-orbits at these places.

Observe first that both $\mathcal{C}$ and $\overline{\mathcal{C}}$ are special biserial algebras, thus we may use the description of the Auslander–Reiten sequences for these algebras given in [WW]. Due to the technical requirements in [WW], we will obtain in this way an explicit description of the convex subquiver spanned by $P_{\mathcal{C}}(r) = P_{\overline{\mathcal{C}}}(r)$ and $I_{\mathcal{C}}(n) = I_{\overline{\mathcal{C}}}(n)$ in $\overline{\mathcal{C}}$-mod rather than in $\mathcal{C}$-mod (but, as we have noted, this does not matter).

Note that both $P_{\mathcal{C}}(r)$ and $I_{\mathcal{C}}(n)$ are so-called *string modules*. Thus in order to describe the corresponding subquiver, we first make a few observations concerning strings and string modules over $\mathcal{C}$ and $\overline{\mathcal{C}}$. The proofs, like all others, will be deferred to the last section.

Recall the definition of a string. Suppose $\mathbf{u} = (u_1, u_2, \ldots, u_t)$ is a sequence of vertices in $C$ such that for every $i = 1, \ldots, t-1$ we have either $u_i \xrightarrow{\varphi} u_{i+1}$ or $u_i \xleftarrow{\psi} u_{i+1}$. We also will assume that no substring is a 0 path in the algebra $\overline{\mathcal{C}}$ (i. e. strings cannot contain Nakayama paths). Such a sequence is called a *string from $u_1$ to $u_t$ over $\overline{\mathcal{C}}$*.

Note that defining strings in this way, we give a direction to the walks in the graph $C$ which correspond to non-zero elements in $\overline{\mathcal{C}}$. Hence a string from $a$ to $b$ is different from a string from $b$ to $a$ for any vertices $a \neq b$ of $C$. With this definition we have the following uniqueness result for strings.

LEMMA 2.1. *For any $a, b \in C$ we have at most one string from $a$ to $b$ over $\overline{\mathcal{C}}$.*

To each string $\mathbf{u}$ over $\overline{\mathcal{C}}$ we can attach a uniquely defined indecomposable $\overline{\mathcal{C}}$-module $M(\mathbf{u})$, called a *string module* (see for example [WW]). Observe that $\mathbf{u} \neq \mathbf{v}$ implies that $M(\mathbf{u}) \not\simeq M(\mathbf{v})$.

The previous lemma shows that we may introduce the following notation: if $a, b \in \overline{B}$ and there exists a string from $a$ to $b$, then the corresponding unique string module will be denoted by $M(a, b)$.

In order to deal with the explicit structure of the Auslander–Reiten quivers of $\mathcal{C}$ and $\overline{\mathcal{C}}$, respectively, we first observe a partial symmetry of the graph $C$.

If $x$ is a vertex of $C$, write $\varphi(x) = y$ and $\varphi^{-1}(y) = x$ provided that there is a $\varphi$-arrow from $x$ to $y$ in $C$. In particular, $\varphi(x)$ is defined for all vertices in $A^-$ and $\varphi^{-1}(x)$ is defined for all vertices in $A^+$.

Also, we write $\varphi^{\omega}(x) = y$ provided the longest $\varphi$-path (possibly of length 0) which starts in $x$ and which does not include a Nakayama path, ends in $y$. Similarly, we write $\varphi^{-\omega}(y) = x$ provided the longest $\varphi$-path which ends in $y$ and wich does not include a Nakayama path, starts in $x$. Note that these functions $\varphi^{\omega}$ and $\varphi^{-\omega}$ are defined for all vertices of $C$. They are not inverse to each other, but they do satisfy the relations $\varphi^{\omega}\varphi^{-\omega}\varphi^{\omega} = \varphi^{\omega}$ and $\varphi^{-\omega}\varphi^{\omega}\varphi^{-\omega} = \varphi^{-\omega}$. It is clear that the image of $\varphi^{\omega}$ is the set of vertices of $B = B^+ = A^0 \cup A^+$, that of $\varphi^{-\omega}$ is the set of vertices of $B^- = A^0 \cup A^-$ and it follows that the restriction of $\varphi^{\omega}$ is a bijection from the set of vertices of $B^-$ to the set of vertices of $B^+$, with the inverse map being $\varphi^{-\omega}$. To simplify the notation, we write $i^*$ for $\varphi^{-\omega}(i)$ for any vertex $i$ of $B^+$. Thus for example, $1^* = r$, where $r$ is the root of the binary tree $B = B^+$.

We deal with the $\psi$-arrows similarly: If there is a $\psi$-arrow in $C$ from $x$ to $y$, then we write $\psi(x) = y$ and $\psi^{-1}(y) = x$. Also, we define $\psi^{\omega}$ and $\psi^{-\omega}$ as above: we put $\psi^{\omega}(x) = y$, provided the longest $\psi$-path which starts in $x$ and which does not include a Nakayama path, ends in $y$, and $\psi^{-\omega}(y) = x$ provided the longest $\psi$-path which ends in $y$ and which does not include a Nakayama path, starts in $x$.

Note that, for any vertex of $C$, there is at most one $\varphi$-arrow and one $\psi$-arrow which starts at this vertex and at most one of each type which ends at this vertex. Thus the above maps and partial maps are well-defined.

Since the vertices of $B$ are indexed by the numbers from 1 to $n$, they may be considered as being totally ordered. Actually, this ordering may be recovered from the binary tree structure of $B$. Using the correspondence $\varphi^{-\omega}$ between the vertices of $B$ and $B^-$ we get a similar total ordering on the vertices of $B^-$. The next lemma shows that this ordering, given by $(1^*, 2^*, \ldots, n^*)$, coincides with the ordering determined by the binary tree structure of $B^-$, provided we switch the role of the $\varphi$- and $\psi$-arrows.

LEMMA 2.2. *For $i = 1, \ldots, n-1$ we have:*

$$i+1 = \begin{cases} \varphi^{\omega}\psi^{-1}(i) & \text{if } i \in \psi(C), \\ \varphi^{-1}\psi^{\omega}(i) & \text{if } i \notin \psi(B^+); \end{cases}$$

$$(i+1)^* = \begin{cases} \psi^{\omega}\varphi^{-1}(i^*) & \text{if } i^* \in \varphi(C), \\ \psi^{-1}\varphi^{\omega}(i^*) & \text{if } i^* \notin \varphi(B^-). \end{cases}$$

Note that the cases for $i+1$ and for $(i+1)^*$ are not disjoint; if both conditions are satisfied, then the two expressions will coincide.

Using this notation and the duality between $B^+$ and $B^-$, we can completely describe the subquiver under consideration.

PROPOSITION 2.3. *The string modules $M_{i,j} = M(i^*, j)$ exist for every pair of indices $1 \le i \le j \le n$, and together they form a convex subquiver $U$ of the Auslander–Reiten quiver of $\overline{\mathcal{C}}$. This subquiver $U$ is isomorphic to $W(n)$, with $M_{i,j}$ corresponding to the point $(i,j)$ in the standard embedding of $W(n)$. This is also an isomorphism of translation quivers when the vertices of $U$ are considered as $\mathcal{C}$-modules.*

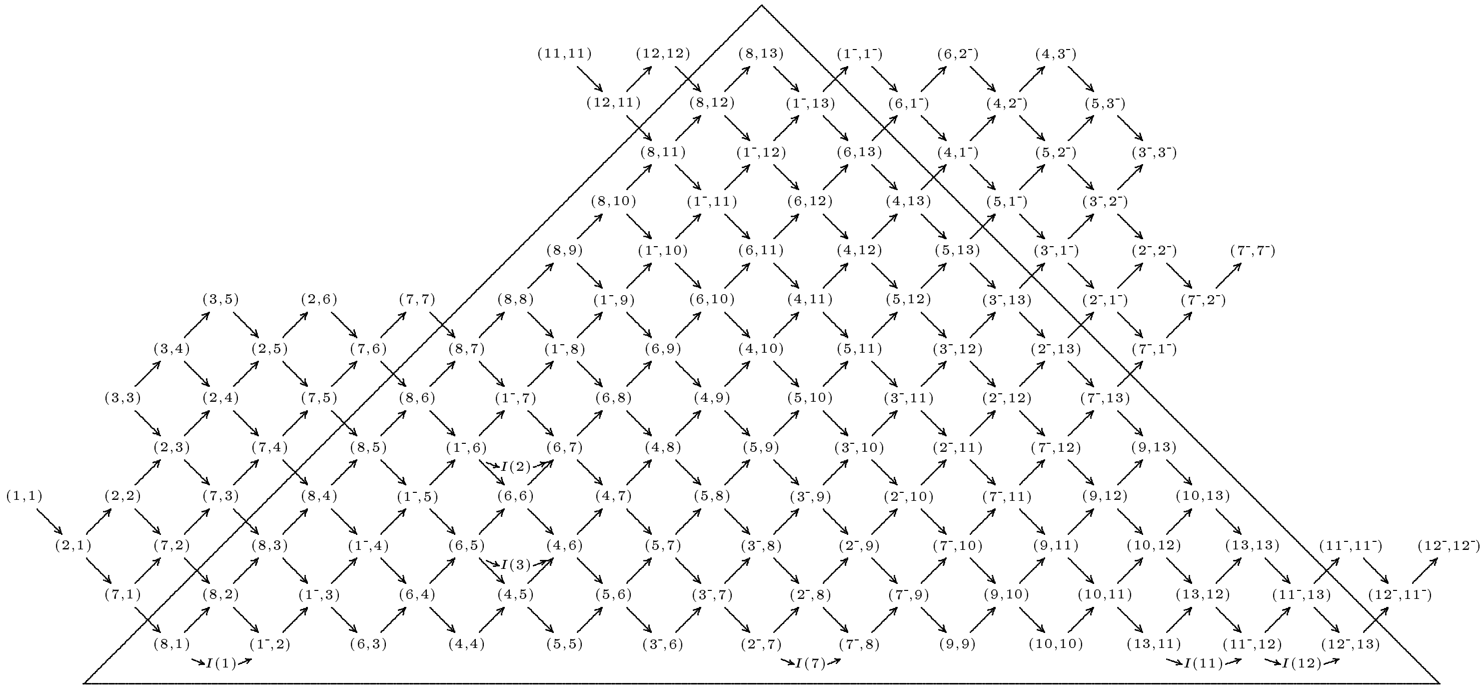
In order to conclude the proof of Theorem 1, we will need only the following statement.

LEMMA 2.4. *Denote by $\ell(i,j)$ the composition length of the module $M_{i,j}$, defined above. Then $\ell(i,j) = f(i,j)$ for every $(i,j) \in W(n)$.*

We have illustrated these results for the Auslander–Reiten quiver of the algebra $\mathcal{C}$ given by our Example in the figure on the next page.

**The case of a stable tube $T(n)$.** Recall first from [ALR] that for every Frobenius function $f$ on $T(n)$ there is a maximal subwing $U$ of $T(n)$, isomorphic to $W(n)$, so that $f$ is additive on the meshes outside $U$.

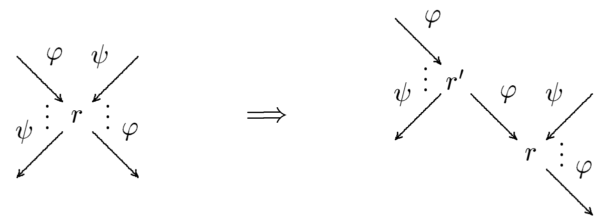
Thus let us take a Frobenius length function $f$ on $T(n)$ and choose a maximal wing, denoted by $W(n)$, containing all the incomplete meshes. We may assume that the restriction of $f$ to the wing is basic, since we will represent our Frobenius function only up to equivalence.




*The Auslander–Reiten quiver of $\mathcal{C}$-mod. Encircled is $U$.*

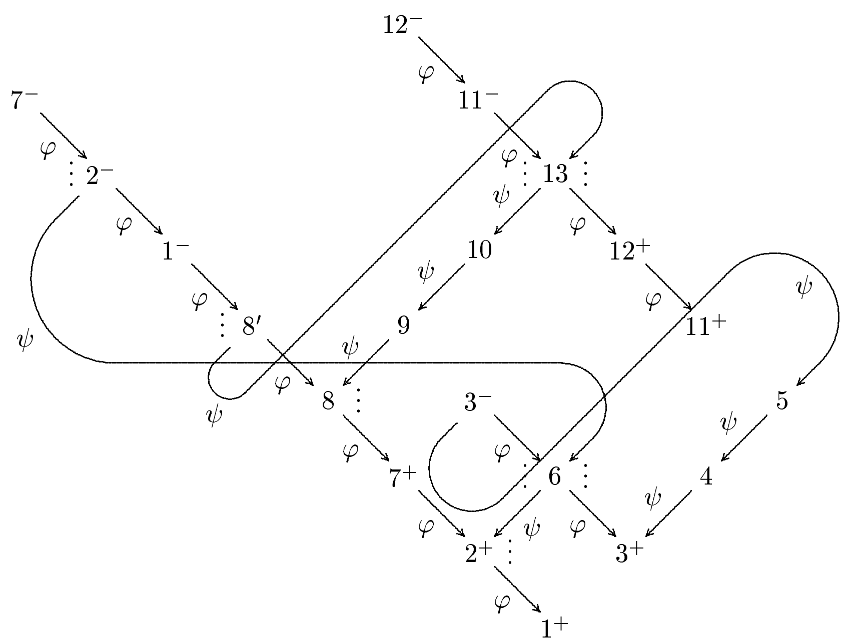
The restriction of $f$ to $W(n)$ will give us a tree $B = B(f)$, as explained above, and we may construct the graph $C$ as before. In order to get a representation of the Frobenius function on the whole of $T(n)$, we have to add to $C$ one $\psi$-arrow, joining the root $r$ of the tree $B(f)$ to the vertex $n$, and then to take the path algebra modulo relations as defined previously (now including also those new relations of the type $\varphi\psi = 0$ or $\psi\varphi = 0$ which arise from adding the new $\psi$-arrow). Unfortunately, by adding this new $\psi$-arrow, we get a cycle in the graph, hence the corresponding algebra will be infinite dimensional and in general it will not have almost split sequences.

In order to obtain a finite dimensional algebra, we can do the following. Replace the root $r$ of the binary tree $B(f)$ by two vertices, $r$ and $r'$, and a $\varphi$-arrow joining $r'$ to $r$ in the following way. The old $\psi$-arrow (if any) in $C$ going to $r$ will now still go to $r$, while the old $\varphi$-arrow will go to $r'$. The old $\varphi$-arrow going from $r$ will now still go from $r$, while the newly added $\psi$-arrow, joining $r$ to the vertex $n$ will go from $r'$. The local situation around $r$ and $r'$ is shown on the following diagram.



We denote the new graph by $D$, and the corresponding algebras by $\mathcal{D}$ and $\overline{\mathcal{D}}$, respectively. As before, $\overline{\mathcal{D}}$ is obtained from $\mathcal{D}$ by factoring out the Nakayama paths for every $i \in A^+$.

From our earlier example we will get the following graph $D$.



We note first that the maps $\varphi^{\omega}$, $\varphi^{-\omega}$, $\psi^{\omega}$ and $\psi^{-\omega}$ are still well defined, and we also keep the notation $i^*$ to denote $\varphi^{-\omega}(i)$ for $1 \le i \le n$. Observe, however, that now $1^* = r'$.

If we consider the strings over $\overline{\mathcal{D}}$, defined analogously to strings over $\overline{\mathcal{C}}$, it is obvious that the uniqueness of strings stated in Lemma 2.1 does not hold any more under the new circumstances. For example, there are many strings from $r$ to $n$: apart from the "straightforward" $\psi^{-1}$-path between them, one may get another path by adding a step from $n$ to $r'$ on the inverse of the newly added $\psi$-arrow, then continuing to $r$ on a $\varphi$-arrow and finally repeating the walk along the $\psi^{-1}$-path from $r$ to $n$. By repeating the procedure we obtain infinitely many such paths.

A crucial fact in this example is that, in order to get new paths, we have to go through the inverse of the new arrow $r' \xrightarrow{\psi} n$ at least once. Thus for a string $\mathbf{u}$, denote by $w(\mathbf{u})$ the number of occurences of this arrow in $\mathbf{u}$; we call it the *winding number of* $\mathbf{u}$. Now we may formulate the following uniqueness statement.

LEMMA 2.5. *For any $a, b \in D$ and any natural number $t \ge 0$, we have at most one string from $a$ to $b$ over $\overline{\mathcal{D}}$ with winding number equal to $t$.*

This allows us to define $M_t(a, b)$ to be the unique string module corresponding to a string from $a$ to $b$ with winding number $t$, provided such a string exists. With this notation we have the following proposition.

PROPOSITION 2.6. *The string modules $M_t(i^*, j)$ exist for every pair of indices $1 \le i \le j \le n$, and $t = 0$ and for every $1 \le i, j \le n$ and $t > 0$. The isomorphism classes of these modules form a component $S$ of the Auslander–Reiten quiver of $\overline{\mathcal{D}}$. This component $S$ is isomorphic, as a quiver, to the underlying quiver of $T(n)$, with $M_t(i, j)$ corresponding to the point $(i, t \cdot n + j)$ in the standard coordinatization of $T(n)$. This is also an isomorphism of translation quivers, when the vertices of $S$ are considered as $\mathcal{D}$-modules.*

Denote by $U$ the maximal wing $\{(i, j) \,|\, 1 \le i \le j \le n\}$ in $S$. Thus $U$ consists of those modules in $S$ whose underlying string has winding number 0.

Note that by "doubling" the vertex $r$, we will not get the original Frobenius function $f$ on $U$, but we will get one equivalent to $f$.

LEMMA 2.7. *Denote by $\ell(i, j)$ the length of the module corresponding to the vertex $(i, j)$ in $S$. Then we have $\ell(1, j) = f(1, j) + 1$ for $1 \le j \le n$. Thus $\ell$ and $f$ are equivalent Frobenius length functions on $U$, and hence they have the same set of incomplete meshes on $S$.*

In this way we have obtained a realization of $f$ on $T(n)$ up to an equivalence, as stated in Theorem 2. If we want to obtain a realization of $f$ itself, we should not insist on working with finite dimensional algebras only.

We consider the case of a positive Frobenius length function on $T(n)$ and assume that $f$ is basic on a maximal wing which contains the incomplete meshes. We go back to the construction of $D$, but keep the root "together", as was originally the case in $B(f)$ and $C$ (i.e. not replacing it with $r' \xrightarrow{\varphi} r$). Denote by $\mathcal{E}$ the corresponding algebra over this quiver with the usual relations. The new arrow $r \xrightarrow{\psi} n$ makes the algebra $\mathcal{E}$ infinite dimensional. Note that the category of (left) $\mathcal{E}$-modules in general will not have Auslander–Reiten sequences. For instance, in our Example there is no almost split sequence starting with the simple module corresponding to the vertex 12.

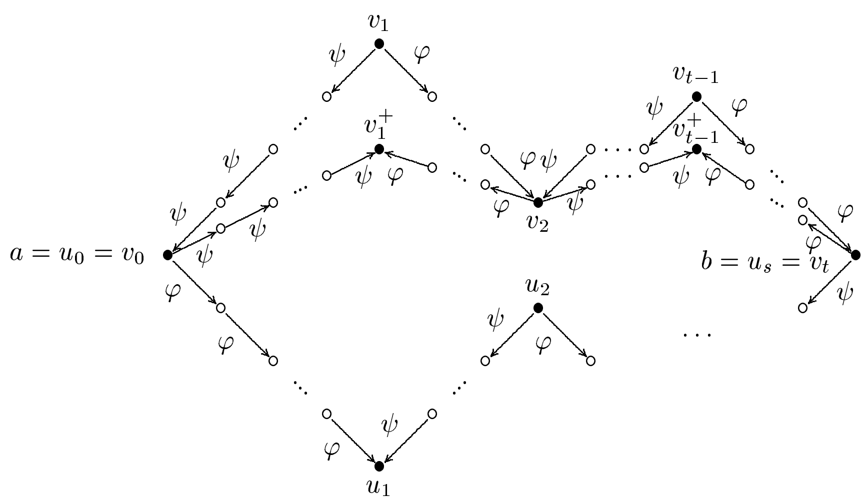
On the other hand, it is clear that the category of (left) $\mathcal{E}$-modules can be identified with the full subcategory of (left) modules $M$ over $\mathcal{D}$ for which multiplication by the arrow $r' \xrightarrow{\varphi} r$ is an isomorphism of the corresponding vector spaces $M_{r'}$ and $M_r$. This subcategory, as indicated, will not have Auslander–Reiten sequences in general. However, it contains the full component $S$, defined in Proposition 2.6, of the Auslander–Reiten quiver of $\mathcal{D}$, and it is clear that $S$ will remain a full component of the "partial" Auslander-Reiten quiver within this subcategory. Furthermore, if we consider the elements of this component as $\mathcal{E}$-modules, the length function on $S$ will coincide with the original Frobenius function $f$, thus giving a realization of the function itself (not only up to equivalence).

## 3. The proofs

*Proof of Lemma* 2.1. Suppose the statement is false and take a minimal counterexample, i.e. consider points $a, b \in C$ such that there are two different strings from $a$ to $b$ and assume that the sum of the lengths of the two strings is minimal. Let $(a = u_0, \ldots, u_1, \ldots \ \ldots, u_s = b)$ and $(a = v_0, \ldots, v_1, \ldots \ \ldots, v_t = b)$ be two such strings, with $(u_i, \ldots, u_{i+1})$ (for $0 \le i \le s-1$) and $(v_j, \ldots, v_{j+1})$ (for $0 \le j \le t-1$) the maximal oriented subpaths of the two strings. As a consequence of the minimality assumption, we may assume that the first string starts with a $\varphi$-arrow, while the second with $\psi^{-1}$.

Since $\psi(C) \subseteq B$, we see that $a \in B$. Now, it follows by induction that $u_{2i-1} \in A^+$ and $u_{2i} \in A^0$ for $i \ge 1$, and it is easy to see that the first string completely belongs to $B$.

The minimality of the strings clearly implies that, if the first string ends with a $\varphi$-arrow, then the second one will have to end in a $\psi^{-1}$-step. This would lead to a contradiction, since then we would have $b \in A^+$, but $\psi$ is not defined on vertices of $A^+$. Hence $s$ and $t$ are even, and $b \in A^0$, furthermore the last arrow of the second string is a $\varphi$-arrow. Now we can use induction once more to show that $v_{t-(2i+1)} \in A^-$ and $v_{t-2i} \in A^0$ for every $i \ge 0$. In particular we find that $a \in A^0$.



Since $v_{2i+1} \in A^-$, we can take the corresponding elements $w_{2i+1} = v_{2i+1}^+ \in A^+$, i. e. we have $v_{2i+1} = w_{2i+1}^-$ for $0 \le i \le t/2$. The construction of $C$ shows that the existence of an oriented $\varphi$-path in $B$ from $c \in B$ to $d \in A^+$ is equivalent to the existence of an oriented $\varphi$-path in $C$ from $d^-$ to $c$. Similarly for the $\psi$-paths. This will give us a nontrivial string from $a$ to $a$ through the elements $a = u_0, u_1, \dots, u_s = v_t, w_{t-1}, v_{t-2}, w_{t-3}, \dots, v_0 = a$ with all the intermediate steps inside $B$. But $B$ is a tree, so we get a contradiction. □

*Proof of Lemma* 2.2. Observe first that the natural ordering of the tree $B$ implies that the following holds for $1 \le i \le n-1$:

$$i+1 = \begin{cases} \varphi^\omega \psi^{-1}(i) & \text{if } i \in \psi(B) \quad \text{(note: } \varphi^\omega\psi^{-1}(i) \text{ is the first element of the } \psi \text{ branch of the subtree whose root is } i \text{ — if there is still such a branch);} \\ \varphi^{-1}\psi^\omega(i) & \text{if } i \notin \psi(B) \quad \text{(note: } \varphi^{-1}\psi^\omega(i) \text{ is the root of the tree whose } \varphi \text{ branch we have just finished).} \end{cases}$$

This establishes the formula for $i+1$ in the case $i \notin \psi(B^+)$. We still have to show that the first formula is also valid if $i \in \psi(C) \setminus \psi(B) = \psi(A^-)$. So let $\psi^{-1}(i) = j^-$; then we get $\varphi^\omega\psi^{-1}(i) = \varphi^\omega(j^-) = \varphi^{-1}\big(\varphi\varphi^\omega(j^-)\big) = \varphi^{-1}(j) = \varphi^{-1}\big(\psi^\omega\psi(j^-)\big) = \varphi^{-1}\psi^\omega(i)$, as required.

Turning now to the formulas for $(i+1)^*$, if $i^* \notin \varphi(B^-)$, then we have $i = \varphi^\omega(i^*) \in A^0$. Since $i < n$, we get from here $i \in \psi(C)$, hence we may use the first formula for $i+1$ to get $i+1 = \varphi^\omega\psi^{-1}(i)$. Since $\psi^{-1}(i) \in B^-$, we get $(i+1)^* = \varphi^{-\omega}\varphi^\omega\psi^{-1}(i) = \psi^{-1}(i) = \psi^{-1}\varphi^\omega(i^*)$. Thus the second case of the formula holds.

As for the first case, note that the condition $i^* \in \varphi(C)$ is equivalent to the condition $i^* = \varphi(i^-)$, where $i \in A^+$. Now, if $i \in \psi(C)$ (and equivalently, $i^- \in \psi^{-1}(C)$), then using the first case of the formula for $i+1$, we get $i+1 = \varphi^\omega\psi^{-1}(i)$, and as in the previous case, $(i+1)^* = \varphi^{-\omega}\varphi^\omega\psi^{-1}(i) = \psi^{-1}(i) = \psi^\omega(i^-) = \psi^\omega\varphi^{-1}(i^*)$, as required. Finally, if $i \notin \psi(C)$, then $i \in A^+$ implies that $i^- \notin \psi^{-1}(C)$, thus, in particular, $\psi^\omega(i^-) = i^-$. Note that $i \in A^+$ also implies that $i \notin \psi^{-1}(C)$, i. e. $\psi^\omega(i) = i$. Hence by the second case of the formula for $i+1$ we get $(i+1)^* = \varphi^{-\omega}(i+1) = \varphi^{-\omega}\big(\varphi^{-1}\psi^\omega(i)\big) = \varphi^{-1}\varphi^{-\omega}(i) = i^- = \psi^\omega(i^-) = \psi^\omega\varphi^{-1}(i^*)$. Thus the first case of the formula for $(i+1)^*$ is fully proved. □

*Proof of Proposition* 2.3. The string from $1^* = r$ to 1 is the $\varphi$-path from the root of $B$ to the vertex 1. Thus $M_{1,1} \simeq P_{\overline{C}}(r)$. We use the characterization of the irreducible maps given in [WW] to show inductively that the other string modules also exist, and that there are irreducible maps going from $M_{i,j}$ to $M_{i+1,j}$ for $1 \le i < j \le n$ and to $M_{i,j+1}$ for $1 \le i \le j < n$.

According to [WW], for a string $(a, \dots, b)$ there are at most two irreducible morphisms going from the module $M(a,b)$: one to a string module $M(a,c)$ and one to a string module $M(d,b)$. We get the corresponding strings by the following algorithm.

(1) If $\big(a, \dots, b, \psi^{-1}(b)\big)$ is a string, then we complete it by $\varphi$-arrows to a string $\big(a, \dots, b, \dots, \varphi^\omega\psi^{-1}(b) = c\big)$. Otherwise let $v$ be the first element in $(a, \dots, b)$ such that the string $(v, \dots, b)$ contains only $\psi^{-1}$-arrows. Then either $a \ne v$, in

which case we choose $c = \varphi^{-1}(v)$, this way obtaining $(a, \dots, c)$ as a substring of $(a, \dots, b) = \big(a, \dots, c, \dots, \psi^{-\omega}\varphi(c) = b\big)$ or $a = v$ and then this type of morphisms does not exist.

(2) If $\big(\varphi^{-1}(a), a, \dots, b\big)$ is a string, then we complete it by $\psi^{-1}$-arrows to the string $\big(d = \psi^{\omega}\varphi^{-1}(a), \dots, a, \dots, b\big)$. Otherwise let $u$ be the last element in $(a, \dots, b)$ such that the string $(a, \dots, u)$ contains only $\varphi$-arrows. Then either $u \neq b$, in which case we choose $d = \psi^{-1}(u)$, this way obtaining $(d, \dots, b)$ as a substring of $(a, \dots, b) = \big(a = \varphi^{-\omega}\psi(d), \dots, d, \dots, b\big)$ or $u = b$ and then this type of morphisms does not exist.

We consider strings of the form $(a, \dots, u, \dots, v, \dots, b)$ where $u$ is the last element of the string such that the substring from $a$ to $u$ is a (possibly empty) $\varphi$-path, while $v$ is the first element of the string such that the substring from $v$ to $b$ is a (possibly empty) $\psi^{-1}$-path.

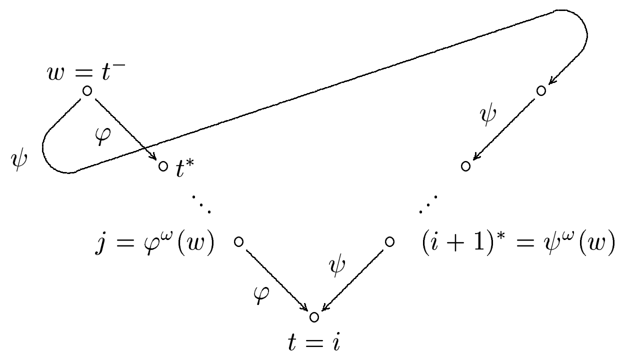
We are going to prove the existence of a morphism of type (1) from $M_{i,j}$ to $M_{i,j+1}$ when $1 \leq i \leq j < n$. If $j \in \psi(C)$ and $(v, \dots, j, \psi^{-1}(j))$ is a string, then the first case of the formula for $j+1$ in Lemma 2.2 (and the uniqueness of strings) shows that we have a string from $i^*$ to $j+1$ of the specified form. Otherwise $j = \psi^{-\omega}(v)$. Consider the directed $\psi^{-1}$-path $\mathbf{p} = (\psi^{\omega}(j), \dots, v, \dots, j)$. Since $j \in B$, we have either $\psi^{\omega}(j) \in A^+$ or $\psi^{\omega}(j) = r$. But $\psi^{\omega}(j) = r$ would imply $j = \psi^{-\omega}(v) = \psi^{-\omega}(r) = n$, contradicting $j < n$. Thus $\psi^{\omega}(j) \in A^+$. If $\mathbf{p}$ is of length 0, then clearly $\psi^{\omega}(j) = v$. Otherwise $\big(\psi^{\omega}(j), \dots, v, \dots, j, \psi^{-1}(j)\big)$ is a Nakayama path, and since $\big(v, \dots, j, \psi^{-1}(j)\big)$ is not a string, $v = \psi^{\omega}(j)$ must hold in this case, too. Now, $v = \psi^{\omega}(j) \in A^+$ implies that $i^* \neq v$, hence $\big(i^*, \dots, \varphi^{-1}(v)\big)$ is a substring of $(i^*, \dots, j)$, and $\varphi^{-1}(v) = \varphi^{-1}\psi^{\omega}(j) = j+1$ by Lemma 2.2. This proves the second case of (1).

Now we assume that $1 \leq i < j \leq n$. We are going to prove the existence of a morphism of type (2) from $M_{i,j}$ to $M_{i+1,j}$. If $i^* \in \varphi(C)$ and $(\varphi^{-1}(i^*), i, \dots, u)$ is a string, then the formula of Lemma 2.2 for $(i+1)^*$ in the case $i^* \in \varphi(C)$ shows that there is a string from $(i+1)^*$ to $j$ of the specified form. Otherwise $i^* = \varphi^{-\omega}(u)$. Observe that $i \neq j$ implies that $\varphi^{-\omega}(u) = i^* \neq j^* = \varphi^{-\omega}(j)$, so $u \neq j$. Hence by the choice of $u$ we find that $u \in \psi(C)$ and hence $u \in B$. But then $u = \varphi^{\omega}\varphi^{-\omega}(u) = \varphi^{\omega}(i^*) = i$ and the new string will go from $\psi^{-1}(u)$ to $j$, and here $\psi^{-1}(u) = \psi^{-1}\varphi^{\omega}(i^*) = (i+1)^*$ by Lemma 2.2. This proves the second case of (2).

Thus we have proved that the modules $M_{i,j}$ $(1 \leq i \leq j \leq n)$ form a subquiver $U$ of the Auslander-Reiten quiver of $\overline{\mathcal{C}}$ isomorphic to the wing $W(n)$. We still have to show that this subquiver is convex. Note that for the modules $M_{i,i}$ $(i = 1, \dots, n)$ the morphisms of type (2) do not exist, since the string $(i^*, \dots, i)$ is a maximal $\varphi$-path, so it can neither be completed nor shortened in the specified way. It is also worth mentioning that there is no morphism of type (1) going into these modules, since these morphisms cannot go to modules corresponding to maximal $\varphi$-strings. The only way to leave the wing is by a morphism of type (1) from a module $M_{i,n}$. But a morphism of type (1) will map a module $M(a, b)$ with $b \in \{n\} \cup A^-$ into a module $M(a, c)$ with $c \in A^-$, since $b \notin \psi(C)$ and in the case when $b = \psi^{-\omega}\varphi(c)$, we find that $\varphi(c) \notin B^+$ and so $c \in A^-$. On the other hand, morphisms of type (2) do not change the end vertex of the string, so once we have left the wing, we cannot get back again by a sequence of irreducible morphisms.

It is clear that knowledge of the irreducible maps determines the translation structure of $U$ for non-projective vertices. Thus to finish the proof, we still have to show that a projective vertex in $U$ which is not projective in $W(n)$ is the endterm of an incomplete mesh from which a projective-injective $\mathcal{C}$-module has been removed. More precisely, we will prove that a string module $M_{i+1,j}$ from $U$ for some $1 \le i < j \le n$ is projective in $\overline{\mathcal{C}}$-mod if and only if there is an arrow $j \xrightarrow{\varphi} i$ in $B$; moreover we will also show that in this case $M_{i+1,j} \simeq P_{\overline{\mathcal{C}}}(i^-)$ in $\overline{\mathcal{C}}$-mod. Thus the mesh ending at this vertex will be completed with the projective-injective module $I_{\mathcal{C}}(i) = P_{\mathcal{C}}(i^-)$ in $\mathcal{C}$-mod.

Assume that $M_{i+1,j}$ is projective as a $\overline{\mathcal{C}}$-module for some $1 \le i < j \le n$. Then the string from $(i+1)^*$ to $j$ must be of the form $\big((i+1)^*, \dots, w, \dots, j\big)$ with $\psi^\omega(w) = (i+1)^*$ and $\varphi^\omega(w) = j$. Clearly, we must have $M_{i+1,j} \simeq P_{\overline{\mathcal{C}}}(w) \in \overline{\mathcal{C}}$-mod. Observe first that $(i+1)^* \ne 1^* = r$, so $\psi^\omega(w) = (i+1)^* \notin A^+$ implies that $w \in A^-$. Thus $w = t^-$ for $t = \varphi\varphi^\omega(t^-) = \varphi\varphi^\omega(w) = \varphi(j) \in A^+$. But then $t^* = \varphi(w)$, therefore Lemma 2.2 implies that $(t+1)^* = \psi^\omega(w) = (i+1)^*$. So $t = i$, hence $M_{i+1,j} \simeq P_{\overline{\mathcal{C}}}(i^-)$ and we also have an arrow $j \xrightarrow{\varphi} i$ in $B$.




Finally, if there is an arrow $j \xrightarrow{\varphi} i$ in $B$, then $i \in A^+$ implies that there is an arrow $i^- \xrightarrow{\varphi} i^*$ in $C$, hence $i^* \in \varphi(C)$. Thus the formula of Lemma 2.2 for $(i+1)^*$ in the case $i^* \in \varphi(C)$ shows that $(i+1)^* = \psi^\omega\varphi^{-1}(i^*) = \psi^\omega(i^-)$, and we also have $j = \varphi^\omega(i^-)$. Since the string from $(i+1)^*$ to $j$ is unique, we get $M_{i+1,j} \simeq P_{\overline{\mathcal{C}}}(i^-)$, as required.

This finishes the proof. □

We should mention that instead of proving the existence of irreducible maps of type (2) between modules $M_{i,j}$ and $M_{i+1,j}$, for $1 \le i < j \le n$, we could first show the existence of these morphisms for the modules $M_{i,i+1}$ and $M_{i+1,i+1}$ — this is a simple special case of the general argument — and then, by induction on the difference $j - i$, show that $\tau M_{i+1,j+1} = M_{i,j}$ for $1 \le i \le j < n$. This would also imply the existence of all required morphisms of type (2).

*Proof of Lemma* 2.4. Proposition 2.3 gives us that the length function $\ell(i,j)$ defined on $W(n)$ is a Frobenius length function with the same set of incomplete meshes as the original Frobenius function $f$. What remains to be shown is that the two functions are actually equal. To this end it is enough to show that the two functions agree on the projective vertices of the wing $W(n)$, since the extension of a basic binary vector to a Frobenius length function on $W(n)$ is unique (cf. [ALR], Lemma 5.1).

Any projective vertex of $W(n)$ corresponds to a string module of the form $M_{1,j} = M(1^*, j)$ for $1 \le j \le n$, and the composition length of this module coincides with the number of vertices in the unique path (string) from the root $1^* = r$ to the vertex $j$ in the binary tree $B = B(f)$. On the other hand, since the given Frobenius function was basic, this is also the value of $f_j = f(\mathbf{p}_j)$.

Thus the composition length of the modules $M_{i,j}$ for $1 \le i \le j \le n$ is $f(i,j)$, as required. □

*Proof of Theorem* 1. By Lemma 2.4 and Proposition 2.3 the algebra $C$ together with the subquiver $U$ gives the required realization. □

*Proof of Lemma* 2.5. It is enough to prove that for any $a, b \in D$ there is at most one string from $a$ to $b$ with winding number equal to 0. Namely, every string $\mathbf{u}$ with winding number equal to $k > 0$ can be partitioned uniquely into substrings as follows: $\mathbf{u} = (u_1, \dots u_{i_1}, u_{i_1+1}, \dots, u_{i_2}, \dots u_{i_k}, u_{i_k+1}, \dots, u_t)$, where $u_{i_j} = n$, $u_{i_j+1} = r'$ for $1 \le j \le k$, furthermore the winding number of the substrings $(u_1, \dots, u_{i_1})$, $(u_{i_1+1}, \dots, u_{i_2})$,..., $(u_{i_k+1}, \dots, u_t)$ is 0. Then, using the uniqueness of the strings with winding number equal 0, we see that the knowledge of the endterms of the string and of the winding number uniquely determines the string.

Thus we can work with strings over the graph $D'$ obtained from $D$ by removing the arrow $r' \xrightarrow{\psi} n$. Furthermore, by collapsing the arrow $r' \xrightarrow{\varphi} r$ to one vertex $r$, any string over $D'$ gives a string over $C$. Two distinct strings over $D'$ from $a$ to $b$ would go to distinct strings over $C$, since clearly, a given string over $D'$ from $a$ to $b$ can be recovered from the collapsed one and the knowledge of the endpoints $a$ and $b$, by replacing $\xrightarrow{\varphi} r$ by $\xrightarrow{\varphi} r' \xrightarrow{\varphi} r$, if necessary. Thus the uniqueness result of Lemma 2.1 gives the required statement. □

*Proof of Proposition* 2.6. We use again the classification of irreducible morphisms by [WW], described earlier in the proof of Proposition 2.3.

First we have to show that for $1 \le i, j \le n$ the following sequence of irreducible morphisms of type (1) exist: $M_0(i^*, i) \to M_0(i^*, i+1) \to \cdots \to M_0(i^*, n) \to M_1(i^*, 1) \to M_1(i^*, 2) \to \cdots \to M_1(i^*, n) \to M_2(i^*, 1) \to \cdots$.

Similarly, for $1 \le i, j \le n$ the following sequence of irreducible morphisms of type (2) exist: $\cdots \to M_2((n-1)^*, j) \to M_2(n^*, j) \to M_1(1^*, j) \to M_1(2^*, j) \to \cdots \to M_1(n^*, j) \to M_0(1^*, j) \to \dots \to M_0((j-1)^*, j) \to M_0(j^*, j)$.

Observe that the existence of string modules of type $M_0(i^*, j)$ for $1 \le i \le j \le n$ and of the specified irreducible morphisms between them follows from the proof of Proposition 2.3. Note first that, having winding number 0, these strings are not effected by the existence of the new arrow $r' \xrightarrow{\psi} n$. Secondly, none of the strings under consideration will start with $r$ or end with $r'$. Thus, the required strings over $\overline{\mathcal{D}}$ can be obtained from the corresponding strings over $\overline{\mathcal{C}}$ by replacing the steps of type $\xrightarrow{\varphi} r$ in the interior of the string with $\xrightarrow{\varphi} r' \xrightarrow{\varphi} r$; if the string starts with $r'$, take the string over $\overline{\mathcal{C}}$ starting with $r$ and add $r' \xrightarrow{\varphi} r$. The morphisms will not be effected because the formulas of Lemma 2.2 are still valid in this new graph $D$.

Next, the existence of the modules $M_t(i^*, j)$ for $t > 0$ and $1 \le i, j \le n$ and of the morphisms $M_t(i^*, j) \to M_t(i^*, j+1)$ for $1 \le j < n$ of type (1) follows from the existence of the modules $M_0(1^*, j)$ and of morphisms $M_0(1^*, j) \to M_0(1^*, j+1)$. This can be seen by writing the corresponding string from $i^*$ to $j$ with winding number $t$ as the concatenation of a string from $i^*$ to $n$, then of $t-1$ copies of the string from $1^* = r'$ to $n$ and finally of the string from $1^*$ to $j$. (The various pieces are joined by the inverse of the arrow $r' \xrightarrow{\psi} n$). Similar considerations show the existence of the specified morphisms $M_t(i^*,j) \to M_t\big((i+1)^*,j\big)$ of type (2) with $1 \le i < n$ and $t > 0$ between modules with winding number $t$.

Finally, we consider the "connecting morphisms" $M_t(i^*,n) \to M_{t+1}(i^*,1)$ of type (1) for $1 \le i \le n$ and $t \ge 0$. As in the previous case, the string corresponding to $M_t(i^*,n)$ can be written as the concatenation of a sring from $i^*$ to $n$ and of $t$ copies of the string from $1^*$ to $n$. Since by adding at the end the vertex $\psi^{-1}(n) = r'$ gives a string (no zero relation is involved), the algorithm implies that the irreducible morphism must go from $M_t(i^*,n)$ to $M_{t+1}\big(i^*,\varphi^{\omega}(r')\big) = M_{t+1}(i^*,1)$, as required. Similarly, the string modules of the form $M_t(n^*,j)$ for $1 \le j \le n$ and $t > 0$ will be mapped to $M_{t-1}(1^*,j)$ by a morphism of type (2), since $n^* \notin \varphi(D)$. Hence one has to cut off the whole substring from $n^*$ to $r'$, thus decreasing the winding number by one.

The fact that $S$ is a full component of the Auslander–Reiten quiver of $\overline{\mathcal{D}}$ can be shown by observing (as in the proof of Proposition 2.3) that there is no irreducible morphism of type (1) going into the modules $M_0(i^*,i)$, and there is no irreducible morphism of type (2) leaving from these modules, since these modules correspond to maximal $\varphi$-strings. Otherwise all the other vertices of $S$ have two incoming and two outgoing arrows.

To finish the proof, we need only note that the proof given for the position of the projective vertices over $\overline{C}$, given in the proof of Proposition 2.3 goes through almost verbatim for this situation. The only thing we still have to check is that neither the modules outside $U$ nor the modules $M_0(1^*,j)$ for $1 \le j \le n$ can be projective over $\overline{\mathcal{D}}$. To this end, note first that all these modules contain a simple composition factor of type corresponding to $r' = 1^*$ in their top. Thus we have to show only that $P_{\overline{\mathcal{D}}}(1^*)$ cannot occur among the modules in $S$. But $P_{\overline{\mathcal{D}}}(1^*) \simeq M_1(r,1)$, and since $r \notin \varphi^{-\omega}(D)$, we are done.

In particular, observe that there are no incomplete meshes outside $U$. □

We remark here that the decomposition of the strings with higher winding number as a concatenation of maximal substrings with winding number equal to 0 corresponds to filtrations of these modules with quotients belonging to $U$. In particular, the module $M_t(i^*,j)$ for $t > 0$ will have a filtration where the first quotient (which is a submodule of $M_t(i^*,j)$) is isomorphic to $M_0(i^*,n)$, then $t-1$ quotients follow, each isomorphic to $M_0(1^*,n)$, and finally the top quotient is isomorphic to $M_0(1^*,j)$.

*Proof of Lemma* 2.7. The length of the string from $1^*$ to $j$ for $1 \le j \le n$ is always greater by 1 than that of the string from $r$ to $j$ (since we start with the arrow $1^* = r' \xrightarrow{\varphi} r$). On the other hand, since we started with a Frobenius fucntion which was basic on $U$, the values $f(1,j)$ for $1 \le j \le n$ are equal to the length of the string from the root $r$ of the tree $B = B(f)$ to $j$. Thus indeed, $\ell(1,j) = f(1,j) + 1$ for $1 \le j \le n$. This implies, in particular, that the sequences $\big(f(1,1),\dots,f(1,n)\big)$ and $\big(\ell(1,1),\dots,\ell(1,n)\big)$ give rise to the same binary tree and hence, by Theorem 1.3 of [ALR], the Frobenius functions $\ell$ and $f$ are equivalent, i.e. they have the same set of incomplete meshes on $S = T(n)$. □

*Proof of Theorem* 2. Lemma 2.7 and Proposition 2.6 show that the algebra $\mathcal{D}$ and the subquiver $S$ of the Auslander–Reiten quiver of $\mathcal{D}$ gives the required realization. □

**Acknowledgment.** The authors gratefully acknowledge the partial support of a German–Hungarian cooperation project for making possible the reciprocal visits during which much of the work was completed. They are indebted to the referree for many helpful suggestions concerning the final presentation of the paper.

## References

[ALR] I. Ágoston, E. Lukács, C.M. Ringel, Frobenius functions on translation quivers, in: *Representation Theory of Algebras*, R. Bautista, R. Martínez-Villa, J.A. de la Peña (editors), CMS Conference Proceedings **18** (1996), 17–37.

[BR] M.C.R. Butler, C.M. Ringel, Auslander–Reiten sequences with few middle terms, with applications to string algebras, *Comm. Algebra* **15** (1987), 145–179.

[HW] D. Hughes, J. Waschbüsch, Trivial extensions of tilted algebras, *Proc. London Math. Soc. (3)*, **46** (1983), 347–364.

[R] C.M. Ringel, *Tame algebras and integral quadratic forms*, Springer Lecture Notes in Mathematics **1099**, 1984.

[WW] B. Wald, J. Waschbüsch, Tame biserial algebras. *J. Algebra* **95** (1985), 480–500.

Mathematical Institute, Hungarian Academy of Sciences, P.O.Box 127, 1364 Budapest, Hungary
*E-mail:* agoston@cs.elte.hu

Department of Algebra, Institute of Mathematics, Technical University of Budapest, P.O.Box 91, 1521 Budapest, Hungary
*E-mail:* lukacs@math.bme.hu

Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld 1, Postfach 100131, Germany
*E-mail:* ringel@mathematik.uni-bielefeld.de