Frobenius functions on translation quivers

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Abstract. Frobenius functions are integral valued functions given on vertices of translation quivers and satisfying certain subadditivity conditions. Typical examples are the length function and the dimension function on the stable Auslander–Reiten quiver of a finite dimensional selfinjective algebra. In our paper we study in detail Frobenius functions on the translation quivers $\mathbb{Z}A_\infty$, $\mathbb{Z}A_\infty^\infty$ and some related ones. In particular we show that there is a one-to-one correspondence between Frobenius functions on the stable tube $T(n)$ and Frobenius functions on the wing $W(n)$, and we classify them using certain related combinatorial structures.

We denote by $W(n)$ the wing of order n (it is the Auslander–Reiten quiver of the ring $A_n(k)$ of upper triangular $n \times n$-matrices over a field k) and by $T(n) = \mathbb{Z}A_\infty/\langle\tau^n\rangle$ the stable tube of rank n (it is the Auslander–Reiten quiver of the locally nilpotent representations of the cyclic quiver with n vertices). Note that $T(n)$ has n full subquivers isomorphic to $W(n)$ which contain all vertices of the mouth; we call them the *maximal wings of* $T(n)$.

Let $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ be a translation quiver (without multiple arrows) and let $f : \Gamma_0 \longrightarrow \mathbb{Z}$ be a map. For any non-projective vertex z of Γ, define $\delta(z) = \delta_f(z) = f(z) + f(\tau z) - \sum\limits_{y \to z} f(y)$. A *Frobenius function* f on Γ is a map $f : \Gamma_0 \longrightarrow \mathbb{Z}$ such that for any non-projective vertex z of Γ with $\delta(z) \neq 0$, we have both $f(z) < \delta(z)$ and $f(\tau z) < \delta(z)$. Note that any non-negative Frobenius function is subadditive, that is, $\delta(z) \geq 0$ for any non-projective vertex. In dealing with a Frobenius function f on Γ, and a non-projective vertex z with $\delta(z) \neq 0$, we say that the mesh ending in z is an *incomplete mesh*. Typical examples of positive Frobenius functions are given by the dimension function or the length function on the stable Auslander–Reiten quiver of a finite dimensional selfinjective algebra; the incomplete meshes being those, where an indecomposable projective module has been removed.

Our aim is a detailed study of positive Frobenius functions on wings, stable tubes and the translation quivers $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty^\infty$. By using certain combinatorial

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structures we give a classification of these functions. In the forthcoming second part of the paper algebraic representations will be given for some of these results. The results were reported at the Sixth International Conference on Representations of Algebras, held at Ottawa in 1992.

We would like to mention that many of the ideas used here can be traced back to [GR]. For unexplained notation, we refer to [R].

1. The main results

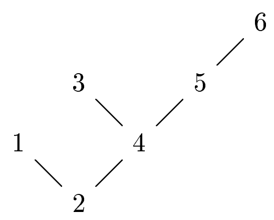
There is a strong interrelation between Frobenius functions on $W(n)$ and $T(n)$.

THEOREM 1.1. *If g is a positive Frobenius function on $T(n)$, then there exists a maximal wing of $T(n)$ which contains all incomplete meshes. Conversely, if U is a maximal wing of $T(n)$ then any function $f : U_0 \longrightarrow \mathbb{N}_1 = \{1, 2, 3, \ldots\}$ has a unique extension to a function $\bar{f} : T(n)_0 \longrightarrow \mathbb{N}_1$ which is additive on all meshes which are not contained in U. Such an extension $\bar{f}$ is a positive Frobenius function if and only if f is.*

Special cases of this statement have been considered by Erdmann (see [E]) in her study of tame symmetric algebras.

A consequence of the above theorem is that any positive Frobenius function on $T(n)$ is uniquely determined by its restriction to an appropriate maximal wing $W(n)$. Thus we may focus our attention to positive Frobenius functions on wings.

Let us denote the projective vertices of $W(n)$ by $\mathbf{p}_1$, $\mathbf{p}_2$, …, $\mathbf{p}_n$, indexed so that there are arrows $\mathbf{p}_1 \to \mathbf{p}_2 \to \cdots \to \mathbf{p}_n$. Given a function $f : W(n) \longrightarrow \mathbb{Z}$, take $f_i = f(\mathbf{p}_i)$ and consider the vector $(f_1, \ldots, f_n)$. We call a vector $(f_1, \ldots, f_n)$ with entries in $\mathbb{N}_1$ *binary* if whenever $f_i = f_j$ for some $i < j$, then there exists an index ℓ such that $i < \ell < j$ and $f_\ell < f_i$. A binary vector $f = (f_1, \ldots, f_n)$ yields an embedded rooted binary tree $B(f)$ with n vertices as follows. The empty vector (for $n = 0$) will correspond to the empty tree. Given a binary vector $f = (f_1, \ldots, f_n)$, let f_t be the minimal coordinate (by definition of a binary vector, t is uniquely defined). Let $g = (f_1 - f_t, \ldots, f_{t-1} - f_t)$, and $h = (f_{t+1} - f_t, \ldots, f_n - f_t)$. We take the vertex with index t as root, and, in case g or h are non-empty, we attach to t the trees $B(g)$ and $B(h)$ so that the root of $B(g)$ is the upper left neighbor and the root of $B(h)$ is the upper right neighbor of t. Note that any embedded rooted binary tree can be obtained in this way. Binary vectors f and f' with $B(f) = B(f')$ will be said to be *equivalent*. For example, the binary vectors $(7, 3, 5, 4, 5, 7)$ and $(2, 1, 3, 2, 3, 4)$ are equivalent, the corresponding binary tree is:



We can now formulate the following theorem about the connection of binary vectors and Frobenius functions on wings.

THEOREM 1.2. *Let f be a positive Frobenius function on the wing $W(n)$, and $f_i = f(\mathbf{p}_i)$. Then $(f_1, \dots, f_n)$ is a binary vector, and every binary vector occurs in this way. Two positive Frobenius functions f and g on $W(n)$ have the same incomplete meshes if and only if the binary vectors $(f_1, \dots, f_n)$ and $(g_1, \dots, g_n)$ are equivalent.*

The binary vectors of length n are just the dimension vectors of (not necessarily basic) tilting modules for $A_n(k)$, see [HR], and therefore part of the previous result, namely that every binary vector arises from Frobenius functions on $W(n)$, follows from [HW]. Here we shall give a direct combinatorial proof.

As one can see from Theorem 1.2, Frobenius functions on $W(n)$ with the same binary tree can be characterized by the set of their incomplete meshes. Actually, not every configuration of meshes in $W(n)$ can occur as the set of incomplete meshes of a Frobenius function. We call two meshes *independent* if the subwings of $W(n)$ generated by these meshes either have at most one vertex in common or one of the subwings contains the other one in its interior. (For a precise definition see Section 4.) A set of meshes is independent if any two meshes in it are independent. We can now state the following theorem.

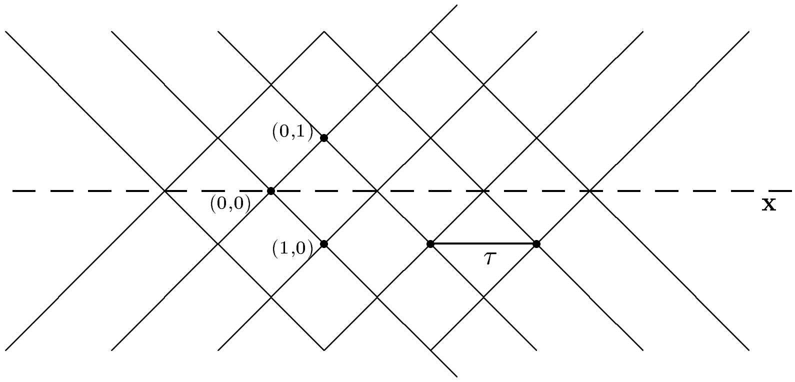
THEOREM 1.3. *A set $\mathcal{S}$ of meshes in the wing $W(n)$ can be obtained as the full set of incomplete meshes of a positive Frobenius function on $W(n)$ if and only if $\mathcal{S}$ is independent.*

Similar characterizations will also be given for the sets of incomplete meshes of Frobenius functions defined on the translation quivers $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty^\infty$.

2. Wings and stable tubes

It seems to be convenient to consider instead of $T(n)$ or $W(n)$ the translation quivers $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty^\infty$.

We shall use the following coordinatization. By definition, the set of vertices of $\mathbb{Z}A_\infty^\infty$ is the set $\mathbb{Z} \times \mathbb{Z}$ of integral lattice points in the plane. There are arrows $(a_1, a_2) \to (a_1 + 1, a_2)$ and $(a_1, a_2) \to (a_1, a_2 + 1)$, and the translation is defined by $\tau(a_1, a_2) = (a_1 - 1, a_2 - 1)$, for any $(a_1, a_2) \in \mathbb{Z} \times \mathbb{Z}$.



Given the lattice points $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (a_1, d_2)$, $\mathbf{c} = (d_1, a_2)$ and $\mathbf{d} = (d_1, d_2)$ with $a_1 < d_1$ and $a_2 < d_2$, we define the *rectangle* $\diamondsuit_{\mathbf{abcd}} = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,|\, a_1 \leq i \leq d_1,\ a_2 \leq j \leq d_2\}$. We shall say that the rectangle $\diamondsuit_{\mathbf{abcd}}$ starts at $\mathbf{a}$ and ends at $\mathbf{d}$. A *mesh* is just a minimal rectangle. The *upper half plane* is the set of points $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,|\, i \leq j\}$, while by the *extended upper half plane* we mean $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,|\, i \leq j+1\}$. (Thus the extended upper half plane will contain also those meshes that have one vertex below the line $\mathbf{x} = \{(i,i) \,|\, i \in \mathbb{Z}\}$.) Lattice points having the same first coordinate will be said to belong to the same *ray*. If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (a_1, b_2)$ are lattice points on the same ray then $(\mathbf{a}, \mathbf{b})$ will denote the set of lattice points $\mathbf{c} = (a_1, c_2)$ where c_2 is an integer between a_2 and b_2 and not equal to any of them. The notation for closed and half open intervals will be also used accordingly. Dually, lattice points with the same second coordinate will be said to belong to the same *co-ray*; we shall use the interval notation for segments of co-rays as well.

Let f be an integer valued function defined on a subset of $\mathbb{Z} \times \mathbb{Z}$. We define the *defect* δ of the function f on an arbitrary rectangle $\diamondsuit_{\mathbf{abcd}}$ in the domain of f by $\delta_{\mathbf{abcd}} = f(\mathbf{a}) + f(\mathbf{d}) - f(\mathbf{b}) - f(\mathbf{c})$. For $\diamondsuit_{\mathbf{abcd}}$ a mesh, we shall also write $\delta(\mathbf{d}) = \delta_{\mathbf{abcd}}$. Given a subset U of $\mathbb{Z} \times \mathbb{Z}$, the function $f : U \to \mathbb{Z}$ is called *additive* (or *subadditive*) if $\delta_{\mathbf{abcd}} = 0$ (or $\delta_{\mathbf{abcd}} \geq 0$, respectively) for every mesh $\diamondsuit_{\mathbf{abcd}}$ in U. A mesh is called *incomplete* (with respect to f) if the defect of the mesh is non-zero. The function f is called a *Frobenius function* if for every incomplete mesh $\diamondsuit_{\mathbf{abcd}}$ we have $\delta_{\mathbf{abcd}} > f(\mathbf{a})$ and $\delta_{\mathbf{abcd}} > f(\mathbf{d})$. If U', U'' are subsets of U, then we write $f(U') < f(U'')$, provided $f(\mathbf{u}') < f(\mathbf{u}'')$, for every $\mathbf{u}' \in U', \mathbf{u}'' \in U''$.

We shall identify $\mathbb{Z}A_\infty$ with the upper half plane, and we call the set of vertices of the form (i,i), with $i \in \mathbb{Z}$ the *boundary* of $\mathbb{Z}A_\infty$. In order to take care of the boundary meshes of $\mathbb{Z}A_\infty$, any function defined on the vertices of $\mathbb{Z}A_\infty$ will be extended to the extended upper half plane by zero. By definition, a *positive Frobenius function on* $\mathbb{Z}A_\infty$ is a Frobenius function f on the extended upper half plane such that $f(i,j) > 0$ for $i \leq j$ and $f(j+1,j) = 0$. We shall consider $\mathbb{Z}A_\infty$ as the universal covering of $T(n)$; in this way each of the vertices of $T(n)$ will correspond to infinitely many lattice points on the upper half plane, and the vertices on the mouth of $T(n)$ correspond to the points on the line $\mathbf{x} = \{(i,i) \mid i \in \mathbb{Z}\}$. Note that a Frobenius function on $T(n)$ lifts to a Frobenius function on $\mathbb{Z}A_\infty$.

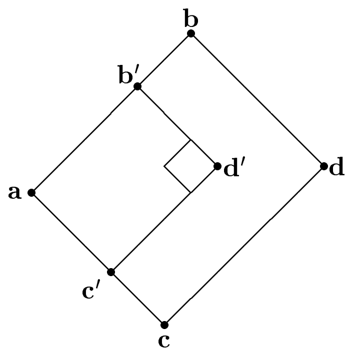
For the wing $W(n)$, we shall fix a standard embedding, mapping the projective vertices $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ to the points $(1,1), (1,2), \ldots, (1,n)$; hence the τ^{-1}-orbit of $\mathbf{p}_1$ will correspond to the points $(1,1), (2,2), \ldots, (n,n)$.

LEMMA 2.1. *Given a function f on a rectangle $\Diamond_{\mathbf{abcd}}$, the defect of f on $\Diamond_{\mathbf{abcd}}$ is the sum of the defects of f on all meshes in $\Diamond_{\mathbf{abcd}}$. Consequently a subadditive function f is additive on $\Diamond_{\mathbf{abcd}}$ if and only if $\delta_{\mathbf{abcd}} = 0$.*

Proof. Straightforward. □

LEMMA 2.2. *Let f be a Frobenius function defined on the rectangle $\Diamond_{\mathbf{abcd}}$ and assume $f((\mathbf{a},\mathbf{c}]) \geq 0$ and $f(\mathbf{a}) \leq f([\mathbf{a},\mathbf{b}])$. Then f is additive on $\Diamond_{\mathbf{abcd}}$.*

Proof. Suppose that f is not additive. Let $\Diamond_{\mathbf{ab'c'd'}}$ be a rectangle in $\Diamond_{\mathbf{abcd}}$ such that it has an incomplete mesh ending at $\mathbf{d}'$ and every other mesh in it is complete.



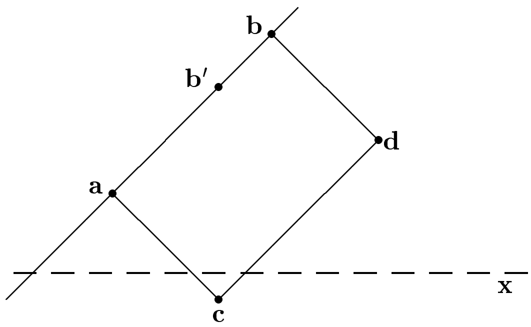
Let us define the function f^* by changing the value of f only at $\mathbf{d}'$ so that the mesh ending at $\mathbf{d}'$ becomes complete. Clearly, $f^*(\mathbf{d}') = f(\mathbf{d}') - \delta(\mathbf{d}') < 0$ because of the Frobenius property. Now f^* is additive on the whole rectangle $\Diamond_{\mathbf{ab'c'd'}}$, on the other hand the defect of f^* is $\delta^*_{\mathbf{ab'c'd'}} = f(\mathbf{a}) - f(\mathbf{b}') - f(\mathbf{c}') + f^*(\mathbf{d}') < 0$, since $f(\mathbf{a}) - f(\mathbf{b}') \leq 0$ and $-f(\mathbf{c}') \leq 0$. This contradicts Lemma 2.1. □

Notice that the above lemma can be dualized, switching the left and right sides of the diagram: $f([\mathbf{c},\mathbf{d})) \geq 0$ and $f(\mathbf{d}) \leq f([\mathbf{b},\mathbf{d}])$ imply that f is additive on $\Diamond_{\mathbf{abcd}}$. The oncoming statements will also admit such dualizations.

LEMMA 2.3. *Let f be a positive Frobenius function on $\mathbb{Z}A_\infty$. Assume that* $\mathbf{a}$ *and* $\mathbf{b}$ *are distinct vertices on the same ray. If $f(\mathbf{a}) = f(\mathbf{b})$ then there exists a point* $\mathbf{b}'$ *in* $(\mathbf{a},\mathbf{b})$ *such that $f(\mathbf{b}') < f(\mathbf{a})$.*

Proof. We may assume that $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (a_1, b_2)$ with $a_2 < b_2$. Suppose the statement is false. Then $f(\mathbf{a}) \leq f([\mathbf{a},\mathbf{b}])$. We apply Lemma 2.2 to the rectangle

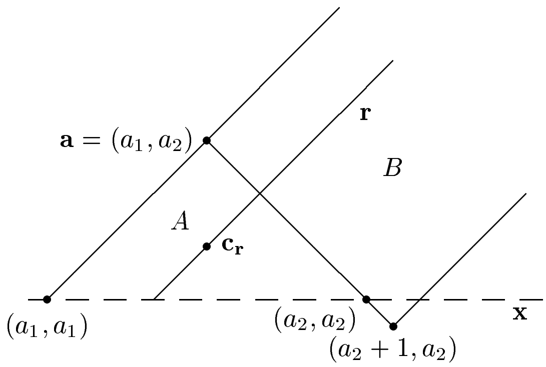
$\Diamond_{\mathbf{abcd}}$ where $\mathbf{c} = (a_2+1, a_2)$ (consequently $f(\mathbf{c}) = 0$) and $\mathbf{d} = (a_2+1, b_2)$ and conclude that f is additive on $\Diamond_{\mathbf{abcd}}$. But then $f(\mathbf{a}) = f(\mathbf{b})$ implies that $f(\mathbf{c}) = f(\mathbf{d})$, whereas $f(\mathbf{c}) = 0$ and $f(\mathbf{d}) > 0$.



This contradiction completes the proof. □

Let f be a positive Frobenius function on $\mathbb{Z}A_\infty$. Assume $\mathbf{a} = (a_1, a_2)$ is a lattice point for which $f(\mathbf{a})$ is the minimal positive value on the ray of $\mathbf{a}$; then $\mathbf{a}$ will be said to be *ray minimal*. Note that according to Lemma 2.3, any ray contains a unique ray minimal vertex in $\mathbb{Z}A_\infty$.

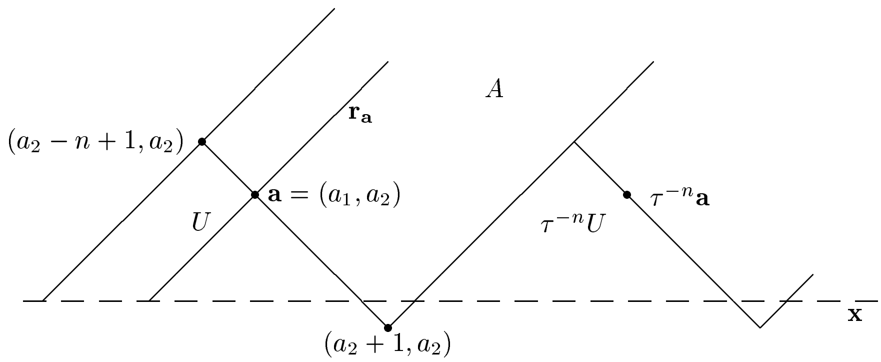
Lemma 2.4. *Let f be a positive Frobenius function on $\mathbb{Z}A_\infty$ and let $\mathbf{a} = (a_1, a_2)$ be a ray minimal lattice point. Then f is additive on the region $B = \{(i,j) \in \mathbb{Z}\times\mathbb{Z} \mid a_1 \le i \le a_2+1,\ a_2 \le j\}$. Moreover, if $\mathbf{r} = \{(i,j) \in \mathbb{Z}\times\mathbb{Z} \mid j \in \mathbb{Z}\}$ is a ray intersecting the region $A = \{(i,j) \in \mathbb{Z}\times\mathbb{Z} \mid a_1 \le i \le j \le a_2\}$, then the unique ray minimal point $\mathbf{c_r} = (i, c_2)$ of $\mathbf{r}$ must belong to A.*



Proof. We may apply Lemma 2.2 to B to get the additivity. As a consequence, the ordering on the ray of $\mathbf{a}$, restricted to B, carries over to the other rays in B, thus we get the second assertion. □

Proposition 2.5. *Let f be a positive Frobenius function on the tube $T(n)$. Then there is a maximal wing U of $T(n)$ such that it contains all the incomplete meshes of f.*

Proof. Instead of $T(n)$, we consider its universal covering $\mathbb{Z}A_\infty$, and we lift the function f to a positive Frobenius function on $\mathbb{Z}A_\infty$ which again will be denoted by f. Choose a ray minimal lattice point $\mathbf{a} = (a_1, a_2)$ such that $a_2 - a_1$ is maximal. (Note that $a_2 - a_1$ is just the distance in $T(n)$ of the vertex which corresponds to $\mathbf{a}$, from the mouth of $T(n)$). Since $\tau^{-n}\mathbf{a} = (a_1 + n, a_2 + n)$ is also a ray minimal point, the second assertion of Lemma 2.4 yields that $a_2 - a_1 \leq n - 1$, so we can include $\mathbf{a}$ into a wing $U = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,|\, a_2 - n + 1 \leq i \leq j \leq a_2\}$ of order n as shown in the following diagram:

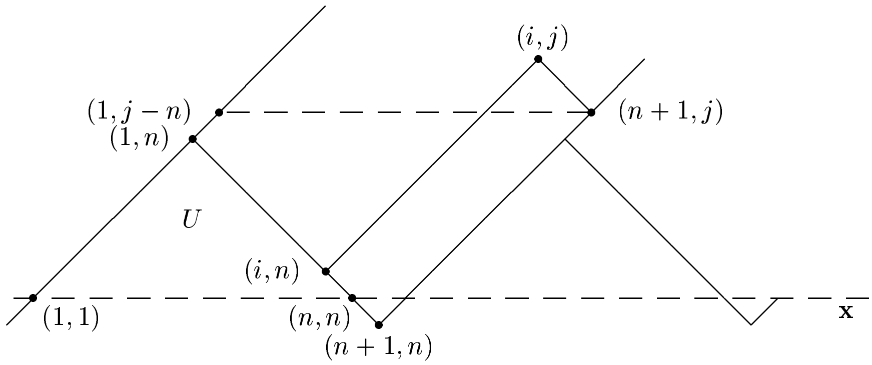


By the choice of the point $\mathbf{a}$ and the wing U, Lemma 2.4 implies that the wing U contains the ray minimal elements of all rays intersecting U. Hence, by using Lemma 2.4 again, we get that the function f is additive on the region $A = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,|\, a_2 - n + 1 \leq i \leq a_2 + 1,\ a_2 \leq j\}$. Since every mesh of $T(n)$ has a representative in $U \cup A$, we have proved the statement. □

Proof of Theorem 1.1. The first assertion is just the statement of Proposition 2.5.

For the converse, we shall again consider $\mathbb{Z}A_\infty$ as the universal covering of $T(n)$. Let us assume that f is defined on the wing $U = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,|\, 1 \leq i \leq j \leq n\}$.

We may define the extension of f to $\{(i,j) \,|\, i = 1, \ldots, n,\ j \geq i\}$ recursively by setting $\bar{f}(i,j) = f(i,n) + \bar{f}(1, j-n)$ for $j > n$.



Simple calculation shows that $\bar{f}$ satisfies the requirements. (Let us note here that by solving the recursion we get:

$$\bar{f}(i,tn+j) = f(i,n) + (t-1)\cdot f(1,n) + f(1,j)$$

for any $t \in \mathbb{N}_1$ and $j \in \{1,2,\dots,n\}$.)

□

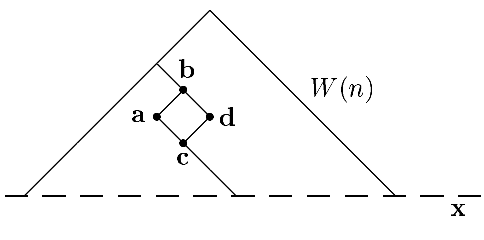
3. Frobenius functions and binary trees

Let us recall that a vector $(f_1, f_2, \dots, f_n)$ with positive integer entries is called *binary* if whenever $f_i = f_j$ with $i < j$ then there is $i < \ell < j$ with $f_\ell < f_i$.

PROPOSITION 3.1. *Let f be a positive Frobenius function on the wing $W(n)$, and $f_i = f(\mathbf{p}_i)$. Then $(f_1, \dots, f_n)$ is a binary vector, and every binary vector occurs in this way.*

Proof. We shall use the standard embedding of $W(n)$ into $\mathbb{Z}A_\infty$. If f is a positive Frobenius function on $W(n)$, and $f_i = f(\mathbf{p}_i)$, then Lemma 2.3 asserts that $(f_1, \dots, f_n)$ is a binary vector.

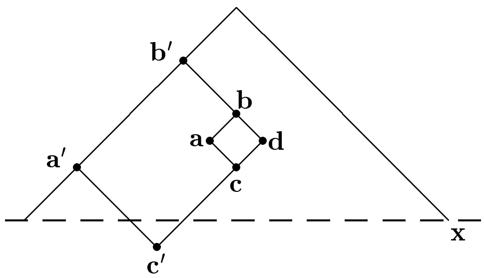
Conversely, assume that a binary vector $(f_1, \dots, f_n)$ is given, and let $f(\mathbf{p}_i) = f_i$. We shall proceed by induction to obtain a positive Frobenius function on $W(n)$. Thus we assume that we have extended the function f to the shaded region in the following diagram and we want to define f on the endpoint $\mathbf{d}$ of the mesh $\diamondsuit_{\mathbf{abcd}}$.



If $f(\mathbf{b})+f(\mathbf{c}) > f(\mathbf{a})$ then define $f(\mathbf{d}) = f(\mathbf{b})+f(\mathbf{c})-f(\mathbf{a})$ so f will be additive on the mesh $\diamondsuit_{\mathbf{abcd}}$. If $f(\mathbf{b}) + f(\mathbf{c}) < f(\mathbf{a})$ then choose $f(\mathbf{d}) = f(\mathbf{b}) + f(\mathbf{c}) + 1$.

(Note that we could have defined $f(\mathbf{d})$ to be $f(\mathbf{b})+f(\mathbf{c})+m$ where m is any positive integer. Hence the Frobenius extension usually will not be unique.) Thus the mesh $\Diamond_{\mathbf{abcd}}$ will be incomplete, satisfying the Frobenius property.

It remains to consider the case when $f(\mathbf{b})+f(\mathbf{c}) = f(\mathbf{a})$. In this case, we define $f(\mathbf{d}) = 0$. Thus f becomes additive on the mesh $\Diamond_{\mathbf{abcd}}$ (but it is not positive). Let $\mathbf{d} = (d_1, d_2)$, and define $\mathbf{c}' = (d_1, d_1 - 1)$, $\mathbf{a}' = (1, d_1 - 1)$ and $\mathbf{b}' = (1, d_2)$. By the dual of Lemma 2.2 we get that f is additive on the rectangle $\Diamond_{\mathbf{a}'\mathbf{b}'\mathbf{c}'\mathbf{d}}$.

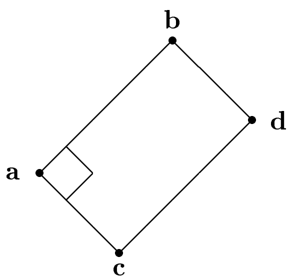


Since $f(\mathbf{c}') = f(\mathbf{d}) = 0 \leq f([\mathbf{c}', \mathbf{d}])$, the additivity implies that $f(\mathbf{a}') = f(\mathbf{b}') \leq f([\mathbf{a}', \mathbf{b}'])$, contradicting the assumption that the starting vector is binary. Thus we see that the case $f(\mathbf{b}) + f(\mathbf{c}) = f(\mathbf{a})$ cannot occur.

The proof is now complete. □

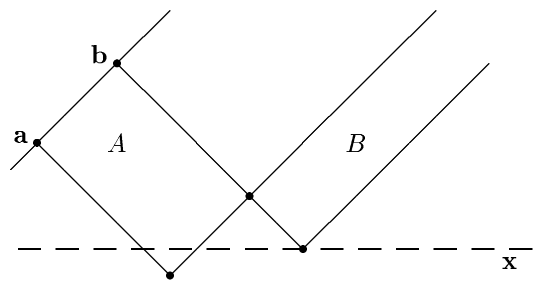
We have seen in the previous proof that the extension of a binary vector to a Frobenius function on the wing $W(n)$ may not be unique. We will show, however, that the position of the incomplete meshes in all these extensions is the same. First we need some further technical remarks.

LEMMA 3.2. *Let f be a non-negative Frobenius function on the rectangle $\Diamond_{\mathbf{abcd}}$ such that the mesh starting at $\mathbf{a}$ is incomplete with respect to f. Then f is additive on the remaining meshes in $\Diamond_{\mathbf{abcd}}$.*



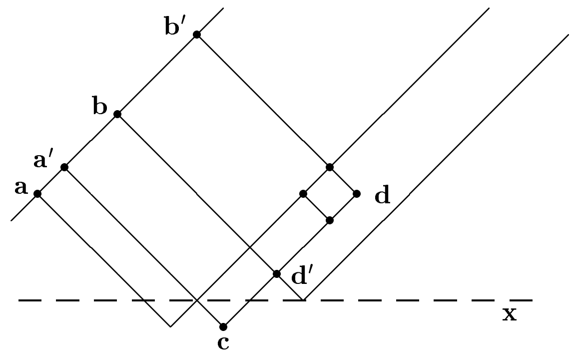
Proof. Let us define a function f^* on $\Diamond_{\mathbf{abcd}}$ by changing the value of f only at $\mathbf{a}$ so that the mesh starting at $\mathbf{a}$ becomes complete with respect to f^*. Thus, by the Frobenius property we have $f^*(\mathbf{a}) < 0$. Then Lemma 2.2 gives that f^* is additive on $\Diamond_{\mathbf{abcd}}$. The statement now follows. □

Lemma 3.3. *Suppose f is a positive Frobenius function on $\mathbb{Z}A_\infty$. Assume that $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (a_1, b_2)$ are lattice points on the same ray with $a_2 < b_2$. If $f(\mathbf{a}),\ f(\mathbf{b}) \leq f((\mathbf{a}, \mathbf{b}))$ and $f(\mathbf{a}) \leq f(\mathbf{b})$ then f is additive in the regions $A = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid a_1 \leq i \leq a_2 + 1,\ a_2 \leq j \leq b_2\}$ and $B = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid a_2 + 1 \leq i \leq b_2,\ b_2 \leq j\}$.*



Proof. Lemma 2.2 implies immediately that f is additive on A.

Suppose now that f is not additive on B. Choose an incomplete mesh ending at $\mathbf{d} = (d_1, d_2)$ (see the shaded part on the diagram below), and define $\mathbf{c} = (d_1, d_1 - 1)$, $\mathbf{a}' = (a_1, d_1 - 1)$, $\mathbf{b}' = (a_1, d_2)$ and $\mathbf{d}' = (d_1, b_2)$.



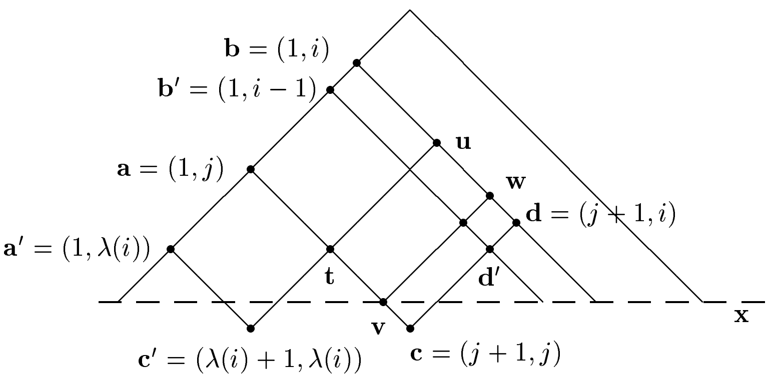
The dual of Lemma 3.2 implies that apart from the mesh ending at $\mathbf{d}$, f is additive on the rectangle $\diamondsuit_{\mathbf{a}'\mathbf{b}'\mathbf{c}\mathbf{d}}$. In particular, f is additive on $\diamondsuit_{\mathbf{a}'\mathbf{b}\mathbf{c}\mathbf{d}'}$. This and the initial condition $f(\mathbf{a}') \geq f(\mathbf{b})$ imply that $0 = f(\mathbf{c}) \geq f(\mathbf{d}')$, contradicting the positivity of the function f. $\square$

For a given binary vector $(f_1, \ldots, f_n)$ and $i \in \{1, \ldots, n\}$ let us define $\lambda_f(i) = \lambda(i) = \max\{j < i \mid f_j < f_i\}$, where $\max \emptyset = 0$. For $i \geq 2$ and $f_{i-1} > f_i$, we have $\lambda(i) + 1 \leq i - 1$, and we define $\mu(i) \in \{\lambda(i) + 1, \ldots, i - 1\}$ so that $f_{\mu(i)}$ is minimal among the values $f_{\lambda(i)+1}, \ldots, f_{i-1}$ (notice that $\mu(i)$ is well-defined since we consider a binary vector).

PROPOSITION 3.4. *Let f be a positive Frobenius function f on $W(n)$, with corresponding binary vector $(f_1, f_2, \dots, f_n)$. Then the incomplete meshes for f are precisely those ending at a lattice-point of the form $(\mu(i)+1, i)$ with $f_{i-1} > f_i$.*

Proof. First of all, according to Lemma 3.2, given any i with $2 \le i \le n$, there is at most one incomplete mesh ending at $(j+1, i)$ for some j, and according to Lemma 2.2, if $f_{i-1} \le f_i$, then there is none.

Thus, we only have to show that in case $f_{i-1} > f_i$, the mesh ending at $\mathbf{d} = (j+1, i)$ is incomplete, where $j = \mu(i)$. Let us define the following points: $\mathbf{a} = (1, j)$, $\mathbf{b} = (1, i)$, $\mathbf{c} = (j+1, j)$, $\mathbf{a}' = \big(1, \lambda(i)\big)$, $\mathbf{b}' = (1, i-1)$, $\mathbf{c}' = \big(\lambda(i)+1, \lambda(i)\big)$, $\mathbf{d}' = (j+1, i-1)$, $\mathbf{t} = \big(\lambda(i)+1, j\big)$, $\mathbf{u} = \big(\lambda(i)+1, i\big)$, $\mathbf{v} = (j, j)$ and $\mathbf{w} = (j, i)$ (see the diagram).



The function f is clearly not additive on $\Diamond_{\mathbf{abcd}}$, since $\delta_{\mathbf{abcd}} = f(\mathbf{a}) - f(\mathbf{b}) + f(\mathbf{d}) - f(\mathbf{c}) = f_j - f_i + f(\mathbf{d}) - 0 > 0$. On the other hand, f is additive on $\Diamond_{\mathbf{a'ac't}}$ and $\Diamond_{\mathbf{tuvw}}$ by Lemma 3.3 (vertically striped region on the diagram). Also, f is additive on $\Diamond_{\mathbf{a'bc'u}}$ (horizontally striped region) as well as on $\Diamond_{\mathbf{ab'cd'}}$ (dotted region) by Lemma 2.2. So the only mesh in $\Diamond_{\mathbf{abcd}}$ that can be incomplete is the one ending at $\mathbf{d}$. $\square$

Let us have a closer look at the relationship between binary vectors and binary trees. We may give an inductive reformulation of the definition of binary vectors as follows. The vector of length zero is binary. A vector $(f_1, \dots, f_n)$ of length $n \ge 1$ is binary, if all $f_i \in \mathbb{N}_1$, the minimum of the f_i occurs only once, say $f_t = \min\{\, f_i \mid 1 \le i \le n \,\}$ for a unique t, and $g = (g_1, g_2, \dots, g_{t-1}) = (f_1 - f_t, f_2 - f_t, \dots, f_{t-1} - f_t)$ and $h = (h_{t+1}, \dots, h_n) = (f_{t+1} - f_t, \dots, f_n - f_t)$ are binary vectors. Now to each binary vector $f = (f_1, \dots, f_n)$ we may attach an embedded rooted binary tree $B(f)$ with multiplicities $m = (m_1, \dots, m_n)$ (attached to the vertices) in the following fashion. The empty tree will correspond to the empty vector. Given the vector $f = (f_1, \dots, f_n)$, where $n \ge 1$, with the index t and the vectors g and h defined as above, we construct the following tree $B(f)$ on the index set of f as vertices. The root will be the vertex t, its multiplicity is $m_t = f_t$; and (in case the vectors g or h are not empty) we connect t with the roots of $B(g)$ and $B(h)$ by an edge so that the root of $B(g)$ is the left upper neighbor of t, and the root of $B(h)$ is the right upper neighbor of t. We shall say that the vertices of $B(g)$ are to the left from t, while those of $B(h)$ are to the right. The multiplicities which we shall attach to the vertices different from t are those defined already for the vertices of $B(g)$ and $B(h)$. Sometimes it will be convenient to give an orientation to the edges: then we put an arrow of type φ starting at t and ending at the root of $B(g)$, and an arrow of type ψ ending at t and starting at the root of $B(h)$. The quiver obtained in this way will be denoted by $\vec{B}(f)$. For example, the binary vector $f = (7, 3, 5, 4, 5, 7)$ yields the following tree $\vec{B}(f)$ with multiplicities:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & & 6 \\
 & 3 & & 5 \arrow[ur, dash, "\psi"'] & \\
1 & & 4 \arrow[ul, dash, "\varphi"] \arrow[ur, dash, "\psi"'] & & \\
 & 2 \arrow[ul, dash, "\varphi"] \arrow[ur, dash, "\psi"'] & & &
\end{tikzcd}
\qquad \text{with values:} \qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & & & & 2 \\
 & 1 & & 1 \arrow[ur, dash] & \\
4 & & 1 \arrow[ul, dash] \arrow[ur, dash] & & \\
 & 3 \arrow[ul, dash] \arrow[ur, dash] & & &
\end{tikzcd}$$

The converse procedure, starting from an embedded, rooted binary tree with multiplicities, is again easy. Namely, let us take the usual ordering of the nodes "from left to right", and add up all the multiplicities from the given node to the root; this gives the vector.

Let $f = (f_1, \ldots, f_n)$ be a binary vector. Besides the function $\lambda = \lambda_f$ we shall consider also the dual function $\rho = \rho_f$ defined by $\rho_f(i) = \rho(i) = \min\{\, j > i \mid f_j < f_i \,\}$ (where $\min \emptyset = n+1$). Clearly $0 \le \lambda(i) < i < \rho(i) \le n+1$ for $i = 1, \ldots, n$. The statement of Proposition 3.4 can be rephrased in the following way. Let $1 \le j < i \le n$. The mesh ending at $(j+1, i)$ is incomplete if and only if $\lambda(j) = \lambda(i)$ and $\rho(j) = i$.

Also, it is easy to see that there is an arrow $i \xrightarrow{\varphi} j$ in $\vec{B}(f)$ if and only if $\rho(j) = i$ and $\lambda(j) = \lambda(i)$. And similarly, the existence of an arrow $j \xrightarrow{\psi} i$ is equivalent to the condition that $\lambda(j) = i$ and $\rho(j) = \rho(i)$. Thus, we can describe the position of the incomplete meshes in $W(n)$ in terms of the tree structure.

PROPOSITION 3.4$'$. *Let f be a positive Frobenius function on $W(n)$, with corresponding binary vector $(f_1, f_2, \ldots, f_n)$. Then the mesh ending at the lattice point $(j+1, i)$ is incomplete with respect to f if and only if there is an arrow $i \xrightarrow{\varphi} j$ in the binary tree $\vec{B}(f_1, \ldots, f_n)$.*

Actually the following proposition shows that the position of the incomplete meshes, the functions λ and ρ attached to the binary vector and the binary tree (without multiplicities) mutually determine each other.

PROPOSITION 3.5 *Let $f = (f_1, \dots, f_n)$ and $f' = (f'_1, \dots, f'_n)$ be two binary vectors, $\vec{B}(f)$ and $\vec{B}(f')$ the corresponding binary trees, and $\lambda = \lambda_f$, $\rho = \rho_f$, $\lambda' = \lambda_{f'}$ and $\rho' = \rho_{f'}$. Then the following statements are equivalent:*

- *(i)* $\lambda = \lambda'$;
- *(i')* $\rho = \rho'$;
- *(ii)* $\vec{B}(f) = \vec{B}(f')$;
- *(iii) for any positive Frobenius extension of f and f' to $W(n)$, the position of the incomplete meshes is the same.*

Proof. The equivalence $(i) \iff (i')$ follows from the formulas $\rho(i) = \min\{\, j \mid i < j \le n,\ \lambda(j) < i \,\}$ and $\lambda(i) = \max\{\, j \mid 1 \le j < i,\ \rho(j) > i \,\}$.

The implication (i) and $(i') \Rightarrow (ii)$ follows immediately from the observation preceding Proposition 3.4′, while $(ii) \Rightarrow (iii)$ is stated in Proposition 3.4′.

To prove $(iii) \Rightarrow (i)$, we have to show how to recover λ from the position of the incomplete meshes. We use induction. Always, we must have $\lambda(1) = 0$. Assume we know $\lambda(j)$ for all $j < i$, where $i \ge 2$. If there is no incomplete mesh ending in a vertex with second coordinate i, then Proposition 3.4 asserts that $f_{i-1} < f_i$, thus $\lambda(i) = i - 1$. Otherwise, there is an incomplete mesh ending in a vertex of the form $(j+1, i)$, where $j < i$. Then by an observation preceding Proposition 3.4′, $\lambda(j) = \lambda(i)$. By induction, $\lambda(j)$ is already known, thus so is $\lambda(i)$. □

Proof of Theorem 1.2. The statements of this theorem are contained in Proposition 3.1 and 3.5. □

4. Configurations of incomplete meshes

Let f be a positive Frobenius function on $\mathbb{Z}A_\infty$. Recall that a vertex $\mathbf{a}$ has been called ray minimal provided $f(\mathbf{a})$ is the minimal positive value on the ray of $\mathbf{a}$. Similarly, a vertex $\mathbf{b}$ may be called *co-ray minimal* provided $f(\mathbf{b})$ is the minimal positive value on the co-ray of $\mathbf{b}$. A vertex which is both ray minimal and co-ray minimal will be said to be *minimal*.

Given a vertex $\mathbf{a} = (a_1, a_2)$ of $\mathbb{Z}A_\infty$ (thus $a_1 \le a_2$), the vertices $\mathbf{b} = (b_1, b_2)$ with $a_1 \le b_1 \le b_2 \le a_2$ will be said to be *dominated by* $\mathbf{a}$; they form a wing $W(\mathbf{a})$ of order $a_2 - a_1 + 1$. This is the wing *generated by* the vertex $\mathbf{a}$. The wing generated by the mesh $\Diamond_{\mathbf{abcd}}$ is the wing $W(\mathbf{b})$.

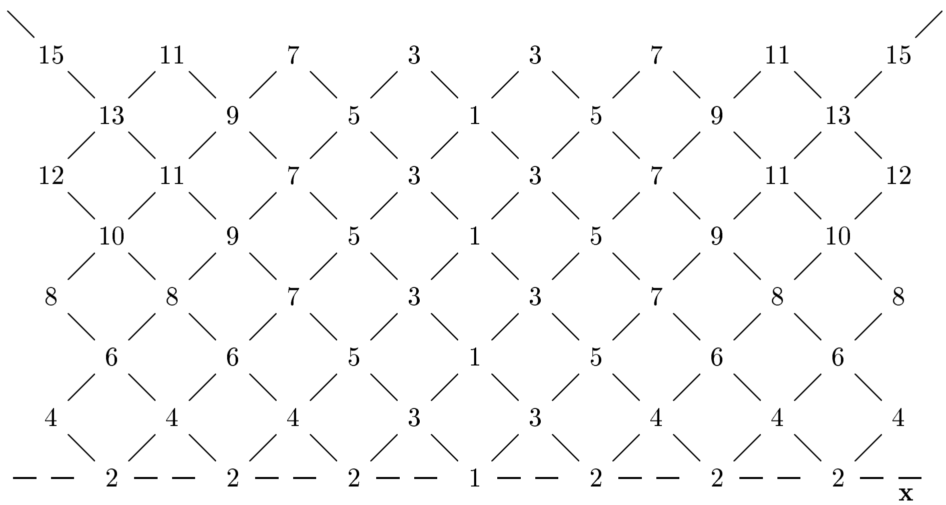
Let f be a positive Frobenius function on $\mathbb{Z}A_\infty$. A set of wings $\mathcal{W} = \{\, W_\ell \mid \ell \in I \,\}$ of $\mathbb{Z}A_\infty$ will be said to be *complete with respect to f* provided any vertex on the boundary of $\mathbb{Z}A_\infty$ belongs to one of the wings, any two elements of $\mathcal{W}$ are either disjoint or one of them contains the other, and finally, if every incomplete mesh is contained in one of the wings in $\mathcal{W}$.

PROPOSITION 4.1. *Let f be a positive Frobenius function on $\mathbb{Z}A_\infty$. The wings $W(\mathbf{a})$, with $\mathbf{a}$ a minimal vertex, form a complete set of wings of $\mathbb{Z}A_\infty$ with respect to f. Conversely, if $\mathcal{W} = \{\, W_\ell \,|\, \ell \in I \,\}$ is a complete set of wings of $\mathbb{Z}A_\infty$, then any minimal vertex $\mathbf{b}$ belongs to some W_ℓ.*

REMARK 4.2. Let $\{\, W_\ell \,|\, \ell \in I \,\}$ be a complete set of wings of $\mathbb{Z}A_\infty$, and let W_ℓ be of order n_ℓ. In case the numbers n_ℓ are bounded, we may select the largest wings with respect to inclusion, and obtain in this way a complete set of pairwise disjoint wings of $\mathbb{Z}A_\infty$. However, the following example shows that there may not exist such a complete set of wings: i.e. we may have to include an increasing chain of wings $W_1 \subset W_2 \subset W_3 \subset \dots$ Let f be the Frobenius function on $\mathbb{Z}A_\infty$, defined by:

$$f((a_1,a_2)) = \begin{cases} 2\,|a_1+a_2|+1 & \text{for } a_1 \le 0 \le a_2, \\ 2(a_2-a_1)+2 & \text{for } 0 < a_1 \le a_2 \text{ or } a_1 \le a_2 < 0. \end{cases}$$

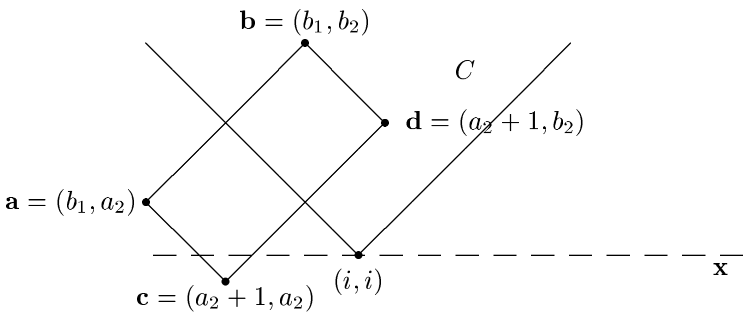
Here precisely those meshes $\Diamond_{\mathbf{abcd}}$ are incomplete for which $\mathbf{b} = (-i,i)$ for some $i \in \mathbb{N}_1$.



To prove Proposition 4.1, we need the following lemmas.

LEMMA 4.3. *Let f be a positive Frobenius function on $\mathbb{Z}A_\infty$. Then any boundary vertex of $\mathbb{Z}A_\infty$ is dominated by a minimal vertex.*

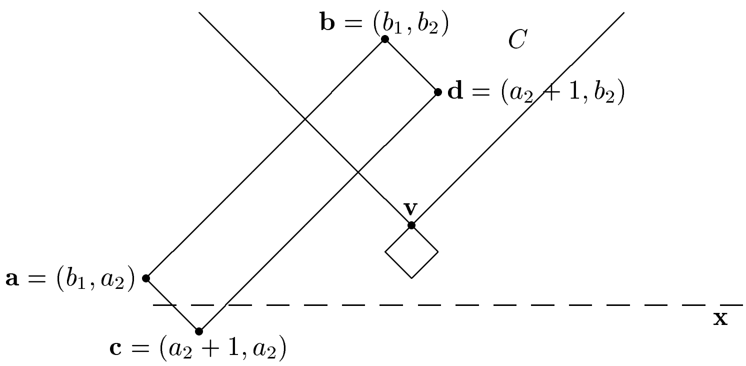
Proof. Consider the vertex (i,i) on the boundary of $\mathbb{Z}A_\infty$, and let C be the set of all vertices (a_1,a_2) with $a_1 \le i \le a_2$. Note that C is the set of all those vertices which dominate the vertex (i,i). Let us choose a vertex $\mathbf{b} = (b_1,b_2) \in C$ such that $f(\mathbf{b}) \le f(C)$. We claim that $\mathbf{b}$ is a minimal vertex. Assume that $\mathbf{b}$ is not ray minimal. Let $\mathbf{a} = (b_1,a_2)$ be ray minimal, thus by assumption $a_2 < i$. Consider the vertices $\mathbf{c} = (a_2+1, a_2), \mathbf{d} = (a_2+1, b_2)$, and apply Lemma 2.2 to the rectangle $\Diamond_{\mathbf{abcd}}$.



We see that f is additive on $\Diamond_{\mathbf{abcd}}$, thus $f(\mathbf{b}) > f(\mathbf{d})$. However, the condition $a_2 < i$ implies that $\mathbf{d}$ belongs to C, thus we obtain a contradiction to the minimality of $f(\mathbf{b})$.

By duality, we also see that $\mathbf{b}$ is co-ray minimal, thus minimal. □

LEMMA 4.4. *Let f be a positive Frobenius function on $\mathbb{Z}A_\infty$. Let $\Diamond_{\mathbf{uvwx}}$ be an incomplete mesh in the extended upper half plane. Then $\mathbf{v}$ is dominated by a minimal vertex.*



Proof. Let $\mathbf{v} = (v_1, v_2)$, and consider the set C of all vertices (a_1, a_2) with $a_1 \leq v_1$, and $v_2 \leq a_2$. Choose a vertex $\mathbf{b} \in C$ such that $f(\mathbf{b}) \leq f(C)$. We claim that $\mathbf{b}$ is a minimal vertex. Assume that $\mathbf{b} = (b_1, b_2)$ is not ray minimal. Let $\mathbf{a} = (b_1, a_2)$ be ray minimal, thus by assumption $a_2 < v_2$. Consider the vertices $\mathbf{c} = (a_2+1, a_2), \mathbf{d} = (a_2+1, b_2)$, and apply Lemma 2.2 to the rectangle $\Diamond_{\mathbf{abcd}}$. We see that f is additive on $\Diamond_{\mathbf{abcd}}$, thus $f(\mathbf{b}) > f(\mathbf{d})$. Since f is additive on $\Diamond_{\mathbf{abcd}}$, the mesh $\Diamond_{\mathbf{uvwx}}$ cannot be inside of $\Diamond_{\mathbf{abcd}}$, thus we must have $a_2 + 1 \leq v_1$, and therefore $\mathbf{d}$ belongs to C. In this way, we obtain a contradiction to the minimality of $f(\mathbf{b})$.

By duality, we also see that $\mathbf{b}$ is co-ray minimal, thus minimal. □

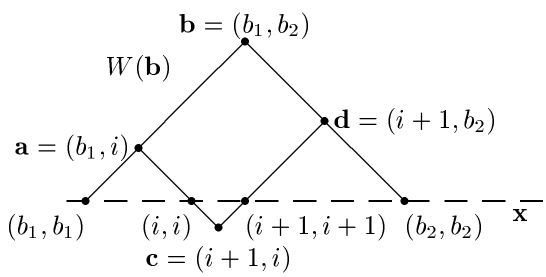
LEMMA 4.5. *Let f be a positive Frobenius function on $\mathbb{Z}A_\infty$. Let $\diamondsuit_{\mathbf{abcd}}$ be a rectangle in the upper half plane. Then at most one of the vertices* $\mathbf{a}$, $\mathbf{d}$ *can be ray minimal.*

Proof. If both $\mathbf{a}$ and $\mathbf{d}$ would be ray minimal then the defect of the rectangle $\diamondsuit_{\mathbf{abcd}}$ would be $\delta_{\mathbf{abcd}} = f(\mathbf{a}) - f(\mathbf{b}) + f(\mathbf{d}) - f(\mathbf{c}) < 0$, contradicting the subadditivity of f. □

Proof of Proposition 4.1. According to Lemma 4.3 and Lemma 4.4, every vertex of the boundary and every incomplete mesh is contained in a wing $W(\mathbf{a})$ for some minimal $\mathbf{a}$.

Let $\mathbf{a}$, $\mathbf{b}$ be minimal vertices, dominating the wings $W(\mathbf{a})$ and $W(\mathbf{b})$, respectively. According to Lemma 4.5, we see that in case these wings intersect, one of them has to be contained in the other one. Thus we conclude that the set of wings $W(\mathbf{a})$ with $\mathbf{a}$ minimal is a complete set.

For the converse, assume there is given a complete set $\mathcal{W} = \{ W_\ell \mid \ell \in I \}$ of wings of $\mathbb{Z}A_\infty$ and take a minimal vertex $\mathbf{b} = (b_1, b_2)$. Consider the wing $W(\mathbf{b})$.



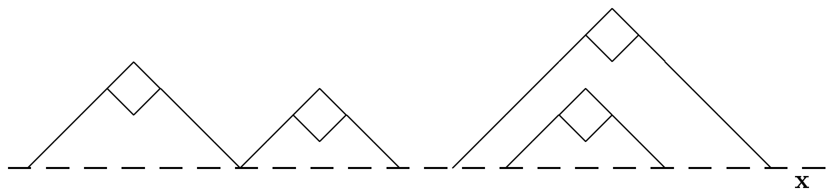
If the vertices (i, i) for $b_1 \leq i \leq b_2$ on the mouth of $W(\mathbf{b})$ are all contained in the same wing $W_\ell \in \mathcal{W}$ then $W(\mathbf{b}) \subseteq W_\ell$, hence $\mathbf{b} \in W_\ell$. On the other hand, if we assume that no single element of $\mathcal{W}$ contains all the vertices from the mouth of $W(\mathbf{b})$, then there is an index $b_1 \leq i < b_2$ such that (i, i) and $(i+1, i+1)$ belong to two disjoint wings in $\mathcal{W}$, moreover for $\mathbf{a} = (b_1, i)$, $\mathbf{c} = (i+1, i)$ and $\mathbf{d} = (i+1, b_2)$, none of the meshes of the rectangle $\diamondsuit_{\mathbf{abcd}}$ is contained in the elements of $\mathcal{W}$. Hence f is additive on $\diamondsuit_{\mathbf{abcd}}$. But the minimality of $\mathbf{b}$ implies for the defect that $\delta_{\mathbf{abcd}} = f(\mathbf{a}) - f(\mathbf{b}) - 0 + f(\mathbf{d}) > 0$, contradicting the additivity.

The proof is now complete. □

Let $\mathcal{S}$ be a finite subset of meshes in the extended upper half plane of $\mathbb{Z} \times \mathbb{Z}$. The *wing cover* of $\mathcal{S}$ is the smallest wing in $\mathbb{Z}A_\infty$ which contains the top vertices of all meshes in $\mathcal{S}$.

In accordance with the definition given for independent meshes in $W(n)$, we call a set of meshes $\mathcal{S}$ in $\mathbb{Z}A_\infty$ *independent* if there is no rectangle $\diamondsuit_{\mathbf{abcd}}$ in the extended upper half plane such that its meshes at $\mathbf{a}$ and at $\mathbf{d}$ are two distinct elements of $\mathcal{S}$. Note that this is equivalent to saying that the distinct meshes

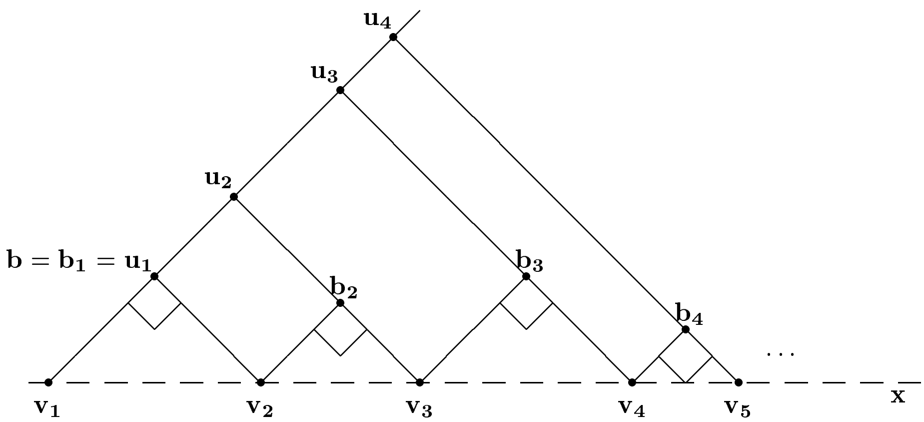
$\Diamond_{\mathbf{abcd}}$ and $\Diamond_{\mathbf{a'b'c'd'}}$ are independent if and only if either $|W(\mathbf{b}) \cap W(\mathbf{b}')| \leq 1$ or $W(\mathbf{b}) \subseteq W(\mathbf{c}')$ or $W(\mathbf{b}') \subseteq W(\mathbf{c})$ (where $W(\mathbf{c})$ and $W(\mathbf{c}')$ may be empty). In case $|W(\mathbf{b}) \cap W(\mathbf{b}')| = 1$, we call the meshes *close*. Two meshes are *$\mathcal{S}$-equivalent* if they are related by the transitive closure of the closeness relation in $\mathcal{S}$.



We may now formulate the following description of the possible positions of the incomplete meshes in $\mathbb{Z}A_\infty$.

PROPOSITION 4.6. *Let $\mathcal{S}$ be a set of meshes in $\mathbb{Z}A_\infty$. Then $\mathcal{S}$ is the full set of incomplete meshes for some positive Frobenius function f on $\mathbb{Z}A_\infty$ if and only if $\mathcal{S}$ is independent and every $\mathcal{S}$-equivalence class of $\mathcal{S}$ is finite.*

Proof. Let $\mathcal{S}$ be the set of incomplete meshes for a Frobenius function f on $\mathbb{Z}A_\infty$. Let us notice first that the independence of $\mathcal{S}$ clearly follows from Lemma 3.2. Suppose now that $\Diamond_{\mathbf{abcd}} \in \mathcal{S}$ is equivalent to infinitely many meshes in $\mathcal{S}$. Clearly, by symmetry we may assume that there exists a sequence of meshes in $\mathcal{S}$ with top vertices $\mathbf{b} = \mathbf{b}_1 < \mathbf{b}_2 < \ldots$ so that the consecutive meshes are close to each other. Let $\mathbf{v}_i$ be the intersection of the line $\mathbf{x}$ with the ray of $\mathbf{b}_i$, while $\mathbf{u}_i$ will denote the intersection of the ray of $\mathbf{b}$ with the co-ray of $\mathbf{b}_i$.

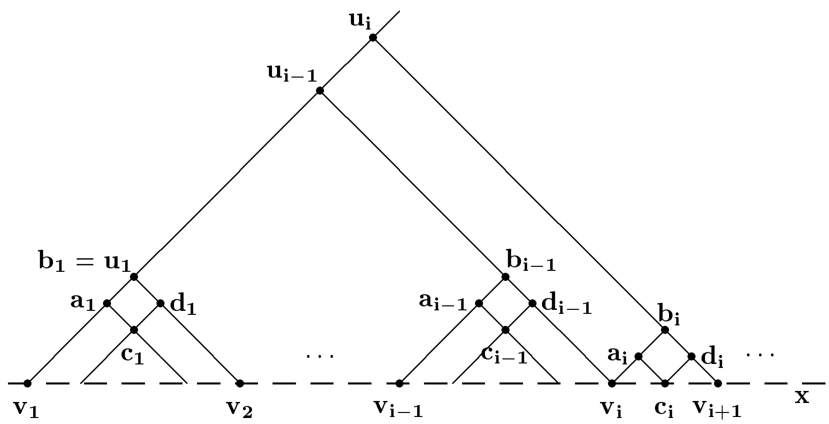


Lemma 2.2 implies that $f(\mathbf{v}_i) > f(\mathbf{b}_i)$ (actually $f(\mathbf{b}_i)$ must be minimal in $f([\mathbf{v}_i, \mathbf{b}_i]))$, thus the additivity of f on $\Diamond_{\mathbf{u_{i-1}u_iv_ib_i}}$ (see the dual of Lemma 3.2) gives that $f(\mathbf{u}_{i-1}) > f(\mathbf{u}_i)$ for $i = 2, 3, \ldots$. Thus we would have an infinite decreasing sequence of values of f, a contradiction.

To prove the other direction, let us take a set $\mathcal{S}$ of meshes, satisfying the conditions. We shall inductively construct a Frobenius function whose incomplete meshes are precisely the prescribed ones. We start with the following observation. An easy argument, similar to the one given in the proof of Theorem 1.1, shows that if we have a Frobenius function on a set of disjoint wings then we can always extend it additively to a wing containing them. In this extension we can freely choose the values of the new vertices on the mouth; the rest is then uniquely determined.

We shall define the function f inductively on the wing covers of the $\mathcal{S}$-equivalence classes. Clearly, if we take two equivalence classes, they either have disjoint wing covers or one of them has an element $\Diamond_{\mathbf{abcd}}$ such that $W(c)$ contains the wing cover of the entire other equivalence class.

Let us now take an $\mathcal{S}$-equivalence class $\mathcal{R} = \{\,\Diamond_{\mathbf{a_i b_i c_i d_i}} \,|\, i = 1, \ldots, n\,\}$ with $\mathbf{b}_1 < \ldots < \mathbf{b}_n$. We define the points $\mathbf{u}_i$ and $\mathbf{v}_i$ as above, completed with $\mathbf{v}_{n+1}$, the intersection of the line $\mathbf{x}$ with the coray of $\mathbf{b}_n$. Let us assume that f is already defined on the wing covers of all other equivalence classes contained in the wing cover of $\mathcal{R}$. According to our earlier observations, we may actually assume that f is defined on each $W(\mathbf{c}_i)$, and the rest is still undetermined. We first extend f to each $W(\mathbf{b}_i)$ by setting $f(\mathbf{v}_i) = n+1$ for every i, extending f from $W(\mathbf{c}_i)$ to $W(\mathbf{a}_i)$ and to $W(\mathbf{d}_i)$ additively, finally setting $f(\mathbf{b}_i) = n$, thus making the mesh $\Diamond_{\mathbf{a_i b_i c_i d_i}}$ incomplete with $\delta_{\mathbf{a_i b_i c_i d_i}} - f(\mathbf{a}_i) = \delta_{\mathbf{a_i b_i c_i d_i}} - f(\mathbf{d}_i) = 1$.



Notice that in this way we have $f([\mathbf{v}_i, \mathbf{b}_i]) \geq n$ and $f([\mathbf{b}_i, \mathbf{v}_{i+1}]) \geq n$ for $i = 1, \ldots, n$. In particular, $f([\mathbf{u}_1, \mathbf{v}_2]) \geq n$. Suppose that f is already extended to $W(\mathbf{u}_{i-1})$ for some $i \leq n$ so that $f([\mathbf{u}_{i-1}, \mathbf{v}_i]) \geq n - i + 2$. Then we can extend f additively to the rectangle $\Diamond_{\mathbf{u_{i-1} u_i v_i b_i}}$, since $f(\mathbf{v}) - f(\mathbf{v}_i) \geq -1$ for every $\mathbf{v} \in [\mathbf{v}_i, \mathbf{b}_i]$ and $f([\mathbf{u}_{i-1}, \mathbf{v}_i]) \geq n - i + 2 \geq 2$. Furthermore, the additivity gives that we have now $f([\mathbf{u}_i, \mathbf{b}_i]) \geq n - i + 2 - 1 = n - i + 1$, while $f([\mathbf{b}_i, \mathbf{v}_{i+1}]) \geq n \geq n - i + 1$, as we observed above.

Thus we can extend f to $W(\mathbf{u}_n)$, i.e. the wing cover of $\mathcal{R}$ so that the incomplete meshes in the new part are exactly the elements of $\mathcal{R}$.

This way we can define f on the wing covers of all $\mathcal{S}$-equivalence classes. If these do not cover the entire half plane then we define arbitrarily the values at the remaining points of the line $\mathbf{x}$, and then extend f additively on the whole $\mathbb{Z}A_\infty$.

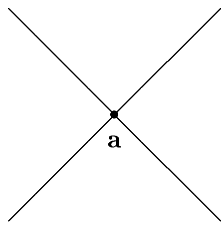
□

Proof of Theorem 1.3. The necessity of independence, as in the case of $\mathbb{Z}A_\infty$, clearly follows from Lemma 3.2. On the other hand, by representing $W(n)$ in $\mathbb{Z}A_\infty$, any subset of meshes of $W(n)$ will have finite $\mathcal{S}$-equivalence classes, hence by Proposition 4.6 there is a Frobenius function on $\mathbb{Z}A_\infty$ with the prescribed set of incomplete meshes. The restriction of this function to $W(n)$ gives us the required representation. □

It turns out that the independence condition is sufficient for the larger quiver $\mathbb{Z}A_\infty^\infty$, too.

PROPOSITION 4.7. *A set $\mathcal{S}$ of meshes in $\mathbb{Z}A_\infty^\infty$ is the set of incomplete meshes for some positive Frobenius function on $\mathbb{Z}A_\infty^\infty$ if and only if $\mathcal{S}$ is independent.*

Proof. The necessity, like before, is obvious from Lemma 3.2. Thus to prove the sufficiency, let us assume that we are given an independent set $\mathcal{S}$ of meshes. Let us define for a vertex $\mathbf{a} = (a_1, a_2)$ the number $n(\mathbf{a})$ to be the number of meshes belonging to $\mathcal{S}$ from the region $A = \{\, \mathbf{b} = (b_1, b_2) \mid b_1 \leq a_1 \text{ and } b_2 \leq a_2 \,\} \cup \{\, \mathbf{c} = (c_1, c_2) \mid c_1 \geq a_1 \text{ and } c_2 \geq a_2 \,\}$ (the shaded region on the diagram).



The independence of $\mathcal{S}$ ensures that this number $n(\mathbf{a})$ is always finite. Now it is easy to show that the function $f(\mathbf{a}) = 2n(\mathbf{a}) + 1$ is a positive Frobenius function such that the incomplete meshes are precisely the elements of $\mathcal{S}$. Namely, for any mesh, when computing the defect, every other mesh is counted the same number of times with plus and with minus sign, while the given mesh itself is counted twice with plus sign only. This gives the additivity for meshes not in $\mathcal{S}$, while for a mesh $\diamondsuit_{\mathbf{abcd}}$ in $\mathcal{S}$ the exact values of f are $f(\mathbf{a}) = f(\mathbf{d}) = 3$ and $f(\mathbf{b}) = f(\mathbf{c}) = 1$. This shows the statement. □

5. Frobenius length functions

The procedure presented in [HR] to associate a binary tree to a basic tilting module for A_n yields for a (not necessarily basic) tilting module T a binary tree with multiplicities, and the corresponding binary vector is just the dimension vector of T.

So we may call a binary vector *basic* provided it is the dimension vector of a basic tilting module, or equivalently, if the corresponding binary tree has all multiplicities equal to 1. It is also easy to see that the binary vector $(f_1, \ldots, f_n)$ is basic if and only if $1 \in \{\, f_i - f_{\lambda(i)}, f_i - f_{\rho(i)} \,\}$ for every $i = 1, \ldots, n$ (where $f_0 = 0 = f_{n+1}$).

It is clear that to any binary vector we may attach its associated basic binary vector by constructing first the corresponding binary tree, changing all the multiplicities to 1 and then taking the binary vector of this basic tree. From Proposition 3.5 it also follows that a binary vector and the associated basic vector will give rise to Frobenius functions on $W(n)$ with the same incomplete meshes.

The importance of basic binary vectors is underlined by the fact that they correspond to the so called Frobenius length functions. A positive Frobenius function f on a subset of $\mathbb{Z} \times \mathbb{Z}$ is called a *Frobenius length function* if for each incomplete mesh $\diamondsuit_{\mathbf{abcd}}$ we have $\delta_{\mathbf{abcd}} - f(\mathbf{a}) = \delta_{\mathbf{abcd}} - f(\mathbf{d}) = 1$. Frobenius length functions on a general translation quiver are defined similarly. Typical examples of Frobenius length functions are the length functions on the components of the stable Auslander–Reiten quiver of a selfinjective algebra.

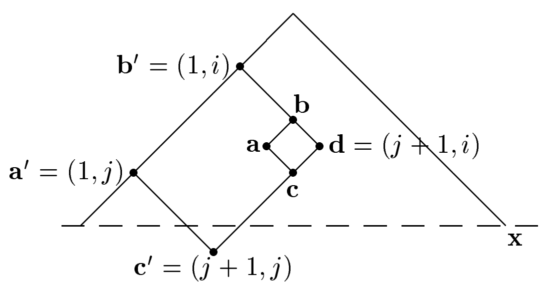
We have the following lemma.

LEMMA 5.1. *A basic binary vector $(f_1, \ldots, f_n)$ has a unique extension f to $W(n)$ that is a Frobenius length function.*

Proof. The construction in the proof of Proposition 3.1 gives a Frobenius extension f for which $\delta_{\mathbf{abcd}} - f(\mathbf{a}) = 1$ for every incomplete mesh $\diamondsuit_{\mathbf{abcd}}$. It also shows that such an extension is unique.

We shall prove now that for the incomplete meshes of any Frobenius extension f of a basic binary vector $\delta_{\mathbf{abcd}} - f(\mathbf{d}) = 1$ must also hold.

Let us take the standard representation of $W(n)$ and assume that $\diamondsuit_{\mathbf{abcd}}$ is an incomplete mesh with $\mathbf{d} = (j+1, i)$. Consider the rectangle $\diamondsuit_{\mathbf{a'b'c'd}}$ with vertices $\mathbf{a'} = (1, j)$, $\mathbf{b'} = (1, i)$ and $\mathbf{c'} = (j+1, j)$.



The dual of Lemma 3.2 implies that $\diamondsuit_{\mathbf{abcd}}$ is the only incomplete mesh in $\diamondsuit_{\mathbf{a'b'c'd}}$, thus by Lemma 2.1 we have $\delta_{\mathbf{abcd}} = \delta_{\mathbf{a'b'c'd}} = f_j - f_i - 0 + f(\mathbf{d})$. According to

Proposition 3.4′ there is an arrow $i \xrightarrow{\varphi} j$ in $\vec{B}(f_1, \ldots, f_n)$, and since $(f_1, \ldots, f_n)$ is basic, we see that $f_j - f_i = 1$. Thus $\delta_{\mathbf{abcd}} = 1 + f(\mathbf{d})$. This finishes the proof. □

Remark 5.2. Note that also non-basic binary vectors may have an extension to $W(n)$ which is a Frobenius length function. In fact, the proof above shows that a binary vector $f = (f_1, \ldots, f_n)$ has such an extension if and only if all the multiplicities at the endpoints of the φ-arrows in $\vec{B}(f)$ are equal to 1 (whereas the remaining multiplicities may be arbitrary).

Corollary 5.3. *Given a positive Frobenius function on $W(n)$, there exists a Frobenius length function on $W(n)$ which has the same set of incomplete meshes.*

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