An extremal problem in the cyclic permutation

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1 Introduction

Let n, k be positive integers and $[n] = \{1, 2, \ldots, n\}$ denote the n-element set. Let $2^{[n]}$ be the power set of $[n]$ and a subset of $2^{[n]}$ is called a *family* of $[n]$. We denote the family of all k-elements subset of $[n]$ by $\binom{[n]}{k}$. A family $\mathcal{F}$ is called *inclusion-free* if for any $F_1, F_2 \in \mathcal{F}$, $F_1 \subsetneq F_2$. As the first theorem in extremal finite set theory, Sperner determined the upper bound of $|\mathcal{F}|$ for inclusion-free families $\mathcal{F}$.

Theorem 1.1 (Sperner theorem)**.** *[4] $max|\mathcal{F}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ where the max is taken over all inclusion-free families.*

For a poset P, we say that a subposet Q' of Q is a (weak) copy of P, if there exists a bijection $f : P \to Q$, such that for any $p, p' \in P$, the relation $p \prec_P p'$ implies $f(p) \prec_Q f(p')$. If a poset Q does not contain a weak copy of P, then it is *P-free*. Specially, a inclusion-free family is a P_2-free poset, where P_2 is the total order on 2 elements.

A family $\mathcal{F}$ is called *intersecting* if for any $F_1, F_2 \in \mathcal{F}$ the intersection $F_1 \cap F_2 \neq \emptyset$. In 1961, Erdős, Ko and Rado gave the upper bound of $|\mathcal{F}|$ for any intersecting family $\mathcal{F}$.

Theorem 1.2. *[1]* $\max |\mathcal{F}| = 2^{n-1}$*, where the* max *is taken over all intersecting families* $\mathcal{F}$*.*

Theorem 1.3. *[1] Let $\mathcal{F}$ be an intersecting family. There is another intersecting family $\mathcal{G}$ such that $\mathcal{F} \subset \mathcal{G}$ and $|\mathcal{G}| = 2^{n-1}$.*

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Theorem 1.4 (Erdős-Ko-Rado Theorem)**.** *[1] Let k (1≤ $k \le \frac{n}{2}$) be a fixed integer. Then* $\max|\mathcal{F}| = \binom{n-1}{k-1}$ *over all intersecting families* $\mathcal{F} \subset \binom{[n]}{k}$

A *cyclic permutation* π of the elements of $[n]$ is an ordering of the elements along a cycle. A subset A of $[n]$ is called an *interval* (along π) if its element are consecutive along π. The following statements are well-known.

Theorem 1.5. *[2] An inclusion-free family* $\mathcal{A}$ *of intervals along* π *has at most* n *elements. If* $\mathcal{A}$ *has* n *elements, then all of its elements have same size.*

Theorem 1.6. *[2] Assume* k *is a positive integer with* $k \le \frac{n}{2}$*. If* $\mathcal{A}$ *is an intersecting family of* k*-element intervals along* π*, then* $|\mathcal{A}| \le k$*.*

In this report, we determine the largest size of a intersecting V-free family of intervals along a fixed permutation π, where the poset $V = \{x, y, z\}$ such that $x \prec y$ and $x \prec z$. It is clear that $|\mathcal{A}| \le n$ if $n \in \{1, 2\}$ and $|\mathcal{A}| \le n+1$ if $n \in \{3, 4\}$, where the upper bound is tight. Thus, we only consider the case $n \ge 5$. Besides, we can assume that π is actually the identity permutation without loss of generality. For general permutaion π, the proof is similar.

Theorem 1.7. *For an intersecting* V*-free family* $\mathcal{A}$ *of intervals along a fixed permutation* π*, if* $n \ge 5$*, then* $|\mathcal{A}| \le \left\lfloor \frac{3}{2}n \right\rfloor$*. In particular, if* $|\mathcal{A}| = \left\lfloor \frac{3}{2}n \right\rfloor$*, then* $|A| > \frac{n}{2}$ *for any* $A \in \mathcal{A}$*.*

Further, let $\mathcal{A}$ be the union of a set of all intervals with size $\left\lfloor \frac{n}{2} \right\rfloor + 1$ and a set of all intervals with size $\left\lfloor \frac{n}{2} \right\rfloor + 2$ and starting point $2i$ $(1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor)$. Then we have $|\mathcal{A}| = \left\lfloor \frac{3}{2}n \right\rfloor$. Thus, the upper bound of Theorem 1.7 is tight.

2 Proof of Theorem 1.7

Without loss of generality, let π be cyclic permutation $(12\ldots n)$. We construct π by drawing a circle with n points, and then label each point from 1 to n in clockwise direction. Then for each proper interval A on π we can denote it by ordering its elements. For $A = i(i+1)\cdots j \pmod{n}$, let i is the starting point of A and j is the end point of A.

For any V-free intersecting family $\mathcal{A}$ of intervals, let $D(\mathcal{A}, \mathcal{E})$ be a directed graph such that $\mathcal{A}$ is the vertex set of D, we have $AB \in \mathcal{E}$ if and only if $A \subset B$ for any two distinct elements A, B of $\mathcal{A}$. In the directed graph, we call a vertex A is a source if and only if $\{B \in \mathcal{A} : BA \in \mathcal{E}\} = \emptyset$. We say A is a sink if and only if $\{B \in \mathcal{A} : AB \in \mathcal{E}\} = \emptyset$. Since $\mathcal{A}$ is V-free, each non-isolated element in $\mathcal{A}$ is either a source or a sink. Let $\mathcal{M}_{\mathcal{A}}$ denote the set of all the sources and isolated elements in $\mathcal{A}$, then obviously $\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$ is the set of sinks in $\mathcal{A}$. We can find that D is a directed bipartite graph $D(\mathcal{M}_{\mathcal{A}}, \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}})$ such that each dipath in D is from $\mathcal{M}_{\mathcal{A}}$ to $\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$. For each element A in $\mathcal{A}$, let

$$deg^{-}(A) = |\{B \in \mathcal{A} : BA \in \mathcal{E}\}|, \quad deg^{+}(A) = |\{B \in \mathcal{A} : AB \in \mathcal{E}\}|.$$

Since $\mathcal{A}$ is V-free, we have $deg^{+}(A) \leq 1$ and $deg^{-}(A) = 0$ for any $A \in \mathcal{M}_{\mathcal{A}}$, $deg^{+}(A) = 0$ and $deg^{-}(A) \geq 1$ for any $A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$. With the property of bipartite graph, we have

$$\sum_{A\in\mathcal{M}_{\mathcal{A}}} deg^{+}(A) = \sum_{B\in\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}} deg^{-}(B). \tag{2.1}$$

We only need to consider the case $|\mathcal{A}| \geq n+1$. It is easy to find such $\mathcal{A}$ with $|\mathcal{A}| = n+1$. For example, $\mathcal{A} = \{A : A \text{ is an interval on } \pi, |A| \geq n-1\}$.

Lemma 2.1. *[2] Assume k is a positive integer with $k \leq \frac{n}{2}$. If $|M| \leq k$ for all $M \in \mathcal{M}_{\mathcal{A}}$, then $|\mathcal{M}_{\mathcal{A}}| \leq k$ and $|\mathcal{A}| \leq 2k$.*

Proof. Denote $|\mathcal{M}_{\mathcal{A}}| = m$ and $\mathcal{M}_{\mathcal{A}} = \{M_1, M_2, \ldots, M_m\}$. Let a_i be the starting point of M_i for $1 \leq i \leq m$, respectively. Since $\mathcal{M}_{\mathcal{A}}$ is inclusion-free, $a_i \neq a_j$ for any i, j. Let M_i' be the interval with starting point a_i and size k. Assume $\mathcal{M}_{\mathcal{A}}' = \{M_1', M_2', \ldots, M_m'\}$. It is easy to see $|\mathcal{M}_{\mathcal{A}}'|$ is intersecting, because $\mathcal{M}_{\mathcal{A}}$ is intersecting. Combining with Theorem 1.6, we have $m = |\mathcal{M}_{\mathcal{A}}| \leq k$. By (2.1), we have

$$k \geq |\mathcal{M}_{\mathcal{A}}| \geq \sum_{A\in\mathcal{M}_{\mathcal{A}}} deg^{+}(A) = \sum_{B\in\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}} deg^{-}(B) \geq |\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}|.$$

So $|\mathcal{A}| = |\mathcal{M}_{\mathcal{A}}| + |\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}| \leq 2k$.

□

Corollary 2.2. *If $|\mathcal{A}| \geq n+1$, then there exists $A \in \mathcal{A}$ such that $|A| > \frac{n}{2}$.*

Since each pair of two distinct elements of $\mathcal{M}_{\mathcal{A}}$ must have different starting points, we have $|\mathcal{M}_{\mathcal{A}}| \leq n$. In the following we consider two cases, $|\mathcal{M}_{\mathcal{A}}| = n$ and $|\mathcal{M}_{\mathcal{A}}| \leq n-1$, to prove $|\mathcal{A}| \leq \lfloor \frac{3}{2}n \rfloor$ provided $|\mathcal{A}| \geq n+1$.

Lemma 2.3. *If $|\mathcal{A}| \geq n+1$ and $|\mathcal{M}_{\mathcal{A}}| = n$, then $|\mathcal{A}| \leq \lfloor \frac{3}{2}n \rfloor$.*

Proof. Let $\mathcal{M}_{\mathcal{A}} = \{M_1, M_2, \ldots, M_n\}$ and M_i has starting point i. Since $\mathcal{M}_{\mathcal{A}}$ is a independent set, we have $|M_i| \leq |M_{i+1}|$ for $i = 1, 2, \cdots, n$ such that we define $M_{n+i} = M_i$. Then we have $|M_1| = |M_2| = \cdots = |M_n|$=k. Combining with Corollary 2.2, we have $k > \frac{n}{2}$. When $k = n$, $\mathcal{A} = \{[n]\}$ which is contradicts with $|\mathcal{A}| \geq n+1$. When $k = n-1$, $\mathcal{A}$ is either $\mathcal{M}_{\mathcal{A}} \cup \{[n]\}$ or $\mathcal{M}_{\mathcal{A}}$, and then $|\mathcal{A}| \leq n+1 \leq \lfloor \frac{3}{2}n \rfloor$, where the last inequality holds because $n \geq 5$.

Recall that for each $A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$, $deg^-(A) \geq 1$. We assume $M_j \in A$. We know that $|A| > |M_j|$, so at least one of $\{M_{j-1}, M_{j+1}\}$ of a subset by A . Thus we have $deg^-(A) \geq 2$ for each $A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$. By (2.1), we have

$$n = |\mathcal{M}_{\mathcal{A}}| \geq \sum_{A\in\mathcal{M}_{\mathcal{A}}} deg^+(A) = \sum_{B\in\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}} deg^-(B) \geq 2|\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}|$$

This implies that $|\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}| \leq \frac{1}{2}|\mathcal{M}_{\mathcal{A}}| = \frac{n}{2}$. Thus, $|\mathcal{A}| = |\mathcal{M}_{\mathcal{A}}| + |\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}| \leq \lfloor \frac{3}{2}n \rfloor$, as desired. □

Lemma 2.4. *If $|\mathcal{A}| \geq n+1$ and $|\mathcal{M}_{\mathcal{A}}| \leq n-1$, then $|\mathcal{A}| \leq \lfloor \frac{3}{2}n \rfloor$ with equality if and only if $|\mathcal{M}_{\mathcal{A}}| = n-1$ and n is odd.*

Proof. If $deg^-(A) \geq 2$ for any $A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$, By (2.1), the result is trival. The only problem is some elements of $\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$ has $deg^- = 1$. Let $T = \{A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}} : deg^-(A) = 1\}$ and $t = |T|$. Obviously, for a fixed $\mathcal{M}_{\mathcal{A}}$, the larger the t, the larger the $|\mathcal{A}|$. We assume t is largest.

Denote $|\mathcal{M}_{\mathcal{A}}| = m$ and $\mathcal{M}_{\mathcal{A}} = \{M_1, M_2, \ldots, M_m\}$. Let a_i be the starting point of M_i for $1 \leq i \leq m$, respectively.

By (2.1), we have

$$m = |\mathcal{M}_{\mathcal{A}}| \geq \sum_{A\in\mathcal{M}_{\mathcal{A}}} deg^+(A) = \sum_{B\in\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}} deg^-(B) \geq t + 2(|\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}| - t)$$

Therefore,

$$|\mathcal{A}| = |\mathcal{M}_{\mathcal{A}}| + |\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}| \leq m + \frac{m+t}{2} \tag{2.2}$$

Let $C = \{a_1, a_2, \ldots, a_m\}$. Suppose that C is the union of d intervals separated by elements not in C. Without loss of generality we can assume that $a_1 = 1$ and n is not in C. The length of j-th interval is denoted by m_j. Then the first interval is $\{a_1, a_1+1, \ldots, a_1+m_1-1\} = \{1, 2, \ldots, m_1\}$. The element a_1+m_1 does not belong to C. The second interval is $\{a_2, a_2+1, \ldots, a_2+m_2-1\}$ and so on. Introduce the following notation for the elements of j-th intervals: $a_{(j,1)}a_{(j,2)}\ldots a_{(j,m_j)}$ $j = 1, 2, \ldots, d$. Recall that $a_{(1,1)} = 1$ and $a_{(d,m_d)} < n$. It is easy to see, $m = \sum\limits_{j=1}^{d} m_j \leq n-d$. Let $M_{(j,i)}$ denote the unique element in $\mathcal{M}_{\mathcal{A}}$ with starting point $a_{(j,i)}$. $\mathcal{M}_{\mathcal{A}p} = \{M_{(j,i)} \in \mathcal{M}_{\mathcal{A}} : j = p\}$. $T_j = \{A \in T : \exists B \in \mathcal{M}_{\mathcal{A}j}, BA \in \mathcal{E}\}$ and $t_j = |T_j|$. Obviously $t = \sum\limits_{j=1}^{d} t_j$. Since $\mathcal{M}_{\mathcal{A}}$ is independent, we have

$$|M_{(j,1)}| \leq |M_{(j,2)}| \leq \cdots \leq |M_{(j,m_j)}|, \text{ for any } j = 1, 2\cdots, d \tag{2.3}$$

$$a_{(j,m_j)} + |M_{(j,m_j)}| \leq a_{(j+1,1)} - 1 + |M_{(j+1,1)}|, j < d \tag{2.4}$$

$$a_{(d,m_d)} + |M_{(d,m_d)}| \leq n + |M_{(1,1)}|, j = d \tag{2.5}$$

For any $A \in T$, assume $M_{(j,i)} \in A$. We know $|A| > |M_{(j,i)}|$ and neither $M_{(j,i-1)}$ or $M_{(j,i+1)}$ is a subset of A. From this and (2.3)(2.4)(2.5), we have:

- if $i = 1$, it always good, i.e $A = M_{(j,i)} \cup \{a_{(j,i)} - 1\}(a \bmod n)$;
- if $2 \leq i \leq m_j - 1$, then $|M_{(j,i+1)}| \geq |M_{(j,i)}| + 1$;
- if $i = m_j$, $j < d$, then $a_{(j+1,1)} - 1 + |M_{(j+1,1)}| \geq a_{(j,m_j)} + |M_{(j,m_j)}| + 1$;
- if $i = m_j$, $j = d$, then n+$|M_{(1,1)}| \geq a_{(d,m_d)} + |M_{(d,m_d)}| + 1$

Since t is the largest and $M_{(j,1)}$ is always good for any $j = 1, 2, \cdots, d$, there always exist $A \in T$ such that $M_{(j,1)} \in A$. and by (2.3)(2.4)(2.5),we have if $j < d$

$$(a_{(j,1)} + m_j - 1) + (t_j - 1 + |M_{(j,1)}|) \leq a_{(j+1,1)} - 1 + |M_{(j+1,1)}|$$

and if $j = d$

$$(a_{(d,1)} + m_d - 1) + (t_d - 1 + |M_{(d,1)}|) \leq n + |M_{(1,1)}|$$

Sum them up, we have

$$\sum_{j=1}^{d} a_{(j,1)} + \sum_{j=1}^{d} m_j + \sum_{j=1}^{d} t_j - d + \sum_{j=1}^{d} |M_{(j,1)}| \leq n + \sum_{j=1}^{d} a_{(j,1)} + \sum_{j=1}^{d} |M_{(j,1)}|$$

Therefore,

$$t \leq n - m + d. \tag{2.6}$$

Computing (2.2), we have $|\mathcal{A}| \leq m + \frac{n+d}{2} \leq \frac{3}{2}n - \frac{d}{2} \leq \lfloor \frac{3}{2}n \rfloor$, where $|\mathcal{A}| = \lfloor \frac{3n}{2} \rfloor$ holds if and only if $d = 1$ and n is odd, as desired. □

Lemma 2.5. *If* $|\mathcal{A}| = \lfloor \frac{3}{2}n \rfloor$, *then* $|A| > \frac{n}{2}$ *for all* $A \in \mathcal{A}$.

Proof. From Lemmas 2.3 and 2.4, we know $|\mathcal{A}| = \lfloor \frac{3}{2}n \rfloor$ if and only if $|\mathcal{M}_{\mathcal{A}}| = n - 1$ and n is odd or $|\mathcal{M}_{\mathcal{A}}| = n$, in particularly, when $|\mathcal{M}_{\mathcal{A}}| = n$ the result is trivial.

When $|\mathcal{M}_{\mathcal{A}}| = n - 1$, n is odd, Let $\mathcal{M}_{\mathcal{A}} = \{M_1, M_2, \ldots, M_{n-1}\}$ and M_i has starting point i. Since $\mathcal{M}_{\mathcal{A}}$ is independent, $|M_i| \leq |M_{i+1}|$ for any $1 \leq i \leq n-2$ and $|M_1| \geq |M_{n-1}| - 1$. Suppose that $|M_1| = k < \frac{n}{2}$. Since $\mathcal{M}_{\mathcal{A}}$ is an intersecting family, we have $k + |M_{k+1}| \geq n+1$. Combining with $|M_{n-1}| \leq |M_1| + 1 = k+1$, we have $n+1-k \leq k+1$ which implies $n \leq 2k < n$, which is impossible. □

3 Extension problem

Theorem 3.1. *For $n \geq 5$, Assume $\mathcal{A}$ is a V-free intersecting family of intervals on π. Then* $\sum\limits_{A\in\mathcal{A}} |A| \leq n(n-2)+\frac{n}{2}(n-1)$.

Proof. For any V-free intersecting family $\mathcal{A}$ of intervals. Let $m = |\mathcal{M}_{\mathcal{A}}|$, $t = |\{A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}} : deg^{-}(A) = 1\}|$.

If $[n] \in \mathcal{A}$, then either $\mathcal{A} = \{[n]\}$ or $\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}} = \{[n]\}$. In the previous case, we have $\sum\limits_{A\in\mathcal{A}} |A| = n$. In the latter case we have

$$\sum_{A\in\mathcal{A}} |A| = n + \sum_{A\in\mathcal{M}_{\mathcal{A}}} |A| \leq n + n(n-1).$$

If $[n] \notin \mathcal{A}$, then $deg^{+}(A) = 0$ for any $A \in \mathcal{M}_{\mathcal{A}}$ with $|A| = n-1$. First, we consider the case $|\mathcal{M}_{\mathcal{A}}| = n$. Since each element of $\mathcal{M}_{\mathcal{A}}$ has same size which is large than $\frac{n}{2}$ and $|\mathcal{A}| \leq \frac{3}{2}n$, we have $\sum\limits_{A\in\mathcal{A}} |A| \leq n(n-2)+\frac{n}{2}(n-1)$ in this case. Now we assume $|\mathcal{M}_{\mathcal{A}}| \neq n$. Let k be the number of elements in $\mathcal{A}$ with size $n-1$. Assume C is the set consisting of starting points of elements of $\mathcal{A}$ and C is separated into d intervals by elements not in C, $d \geq 1$. We have

$$m-k = \sum_{A\in\mathcal{M}_{\mathcal{A}}} deg^{+}(A) = \sum_{B\in\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}} deg^{-}(B) \geq t + 2(|\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}| - t).$$

Therefore, $|\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}| \leq \frac{m-k+t}{2}$.

$$\sum_{A\in\mathcal{A}} |A| = \sum_{A\in\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}} |A| + \sum_{A\in\mathcal{M}_{\mathcal{A}}} |A| \leq (m-k)(n-2) + k(n-1) + \frac{m-k+t}{2}(n-1).$$

By (2.6), we have $\sum\limits_{A\in\mathcal{A}} |A| \leq n(n-2)+\frac{n}{2}(n-1)-(k+d)\frac{n-3}{2} < n(n-2)+\frac{n}{2}(n-1)$. □

Let π_i^j denote the interval on π with starting point i and size j. It is easy to find such $\mathcal{A}$ with $\sum\limits_{A\in\mathcal{A}} |A| = n(n-2)+\frac{n}{2}(n-1)$ in Theorem 3.1. For example, let

$$\mathcal{A} = \bigcup_{i=1}^{n}\{\pi_i^{n-2}\} \cup \bigcup_{i=1}^{\lfloor\frac{n}{2}\rfloor}\{\pi_{2i}^{n-1}\}.$$

So the upper bound of Theorem 3.1 is tight.

Theorem 3.2. *For $n = 2p+1, p \geq 2$, assume $\mathcal{A}$ is a V-free intersecting family of intervals on π. Then* $\sum\limits_{A\in\mathcal{A}} \binom{2p+1}{|A|} \leq (2p+1)\binom{2p+1}{p+1} + p\binom{2p+1}{p+2}$

Proof. Recall that $|\mathcal{M}_{\mathcal{A}}| \leq n = 2p+1$ and $|\mathcal{A}| \leq \frac{3n-1}{2} = 3p+1$. If $|A| \geq p+1$ for any $A \in \mathcal{M}_{\mathcal{A}}$, then we have

$$\sum_{A\in\mathcal{A}} \binom{2p+1}{|A|} = \sum_{A\in\mathcal{M}_{\mathcal{A}}} \binom{2p+1}{|A|} + \sum_{A\in\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}} \binom{2p+1}{|A|} \leq (2p+1)\binom{2p+1}{p+1} + p\binom{2p+1}{p+2}.$$

Now assume that there exists an element of $\mathcal{M}_{\mathcal{A}}$ with size at most p. By Lemma 2.5, this implies $|M_A| \leq 2p$ and $|\mathcal{A}| \leq 3p$. Let $m = |\mathcal{M}_{\mathcal{A}}|$ and $t = |\{A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}} : deg^{-}(A) = 1\}|$. Let C is the set consisting of starting points of elements of $\mathcal{A}$. Suppose that C is the union of d intervals separated by elements not in C.

Let $\mathcal{A}_1 = \{A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}} : |A| \leq p+1\}$ and and $k = |\{B \in \mathcal{M}_{\mathcal{A}} : B \subsetneq A, |B| \leq p\}|$. Combining with Lemma 2.1and $\mathcal{M}_{\mathcal{A}}$ is intersecting, we have

$$p \geq k \geq \sum_{\substack{A\in\mathcal{M}_{\mathcal{A}}\\ |A|\leq p}} deg^{+}(A) \geq \sum_{B\in\mathcal{A}_1} deg^{-}(B) \geq t_1 + 2(|\mathcal{A}_1| - t_1),$$

where $t_1 = |\{A \in \mathcal{A}_1 : deg^{-}(A) = 1\}| \leq |\mathcal{A}_1|$.

Therefore, $|\mathcal{A}_1| \leq \frac{1}{2}(k + t_1)$ implys $|\mathcal{A}_1| \leq k$. Let $\mathcal{M}_1 = \{B \in \mathcal{M}_{\mathcal{A}} : B \subsetneq A, |B| \leq p+1\}$ and $m_1 = |\{B \in \mathcal{M}_{\mathcal{A}} : B \subsetneq A, |B| \leq p+1\}|$. It is easy to see that if $\pi_1^q \in \mathcal{M}_1$ where $q \leq p$, then $\pi_{q+1}^j \notin \mathcal{A}_1$ where $j \leq p+1$. Since $\mathcal{M}_{\mathcal{A}}$ is inclusion-free and M_1 is a subset of $\mathcal{M}_{\mathcal{A}}$. We have $m_1 \leq n - k = 2p+1-k$.

$$\begin{aligned}
\sum_{A\in\mathcal{A}} \binom{2p+1}{|A|} &= \sum_{A\in\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}} \binom{2p+1}{|A|} + \sum_{A\in\mathcal{M}_{\mathcal{A}}} \binom{2p+1}{|A|} \\
&= \sum_{A\in\mathcal{A}-\mathcal{A}_1-\mathcal{M}_1} \binom{2p+1}{|A|} + \sum_{A\in\mathcal{A}_1} \binom{2p+1}{|A|} + \sum_{A\in\mathcal{M}_1} \binom{2p+1}{|A|} \\
&\leq (|\mathcal{A}_1| + m_1)\binom{2p+1}{p+1} + (|\mathcal{A}| - |\mathcal{A}_1| - m_1)\binom{2p+1}{p+2} \\
&\leq (2p+1)\binom{2p+1}{p+1} + (p-1)\binom{2p+1}{p+2},
\end{aligned}$$

where the last equality holds because

- $|\mathcal{A}| \leq 3p$;
- $|\mathcal{A}_1| \leq k$ and the coefficient of $|\mathcal{A}_1|$ is $\binom{2p+1}{p+1} - \binom{2p+1}{p+2}$ which is larger then 0;
- $m_1 \leq 2p+1-k$ and the coefficient of m_1 is $\binom{2p+1}{p+1} - \binom{2p+1}{p+2}$ which is larger then 0;

□

It is easy to find such $\mathcal{A}$ which reach the upper bound in Theorem 3.2. For example, let

$$\mathcal{A} = \bigcup_{i=1}^{2p+1} \{\pi_i^{p+1}\} \cup \bigcup_{i=1}^{p} \{\pi_{2i}^{p+2}\}.$$

So the upper bound of Theorem 3.2 is tight.

Theorem 3.3. *For $n = 2p$ with $p \geq 3$, assume $\mathcal{A}$ is a V-free intersecting family of intervals on π. Then $\sum\limits_{A\in\mathcal{A}} \binom{2p}{|A|} \leq p\binom{2p}{p} + \left\lfloor \frac{3}{2}p \right\rfloor \binom{2p}{p+1} + \left\lfloor \frac{1}{2}(p-1) \right\rfloor \binom{2p}{p+2}$.*

Proof. Let $n = 2p$ with $p \geq 3$. By similiarly argument with Theorem 3.2, if $|A| > p$ for all $A \in \mathcal{M}_{\mathcal{A}}$, then we have $\sum\limits_{A\in\mathcal{A}} \binom{2p}{|A|} \leq 2p\binom{2p}{p+1} + p\binom{2p}{p+2}$. If $|A| \leq p$ for any $A \in \mathcal{M}_{\mathcal{A}}$, by Lemma 2.1, we have $|\mathcal{M}_{\mathcal{A}}| \leq p$ and $\sum\limits_{A\in\mathcal{A}} \binom{2p}{|A|} \leq 2p\binom{2p}{p} \leq 2p\binom{2p}{p+1} + p\binom{2p}{p+2}$.

Now we assume that $\mathcal{M}_{\mathcal{A}}$ contain both an element with at most size p and an element with size at least $p+1$. By Lemma 2.5, this implies $|\mathcal{M}_{\mathcal{A}}| \leq 2p-1$ and $|\mathcal{A}| \leq 3p-1$. Let $m = |\mathcal{M}_{\mathcal{A}}|$ and $t = |\{A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}} : deg^-(A) = 1\}|$. Assume C is the set consisting of starting points of elements of $\mathcal{A}$. Suppose that C is the union of d intervals separated by elements not in C.

Let $\mathcal{A}_1 = \{A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}} : |A| \leq p+1\}$ and $m_1 = |\{B \in \mathcal{M}_{\mathcal{A}} : B \subsetneq A, |B| \leq p\}|$. Combining with Lemma 2.1 and $\mathcal{M}_A$ is intersecting, we have

$$p \geq m_1 \geq \sum_{B\in\mathcal{A}_1} deg^-(B) \geq t_1 + 2(|\mathcal{A}_1| - t_1),$$

where $t_1 = |\{A \in \mathcal{A}_1 : deg^-(A) = 1\}| \leq t$. This implies $|\mathcal{A}_1| \leq \frac{m_1+t}{2}$.

Let $\mathcal{A}_2 = \{A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}} : |A| = p\}$ and $k = |\{B \in \mathcal{M}_{\mathcal{A}} : B \subsetneq A, |B| < p\}|$. Similarly, $m_1 \geq k \geq t_2 + 2(|\mathcal{A}_2| - t_2)$, where $t_2 = |\{A \in \mathcal{A}_2 : deg^-(A) = 1\}|$. Note that $t_2 \leq |\mathcal{A}_2|$. This implies $|\mathcal{A}_2| \leq k$. We have

$$\begin{aligned}
\sum_{A\in\mathcal{M}_{\mathcal{A}}} \binom{2p}{|A|} \leq& (m - m_1 + k)\binom{2p}{p+1} + (m_1 - k)\binom{2p}{p}. \\
\sum_{A\in\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}} \binom{2p}{|A|} \leq& |\mathcal{A}_2|\binom{2p}{p} + (|\mathcal{A}_1| - |\mathcal{A}_2|)\binom{2p}{p+1} + (|\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}| - |\mathcal{A}_1|)\binom{2p}{p+2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{A\in\mathcal{A}} \binom{2p}{|A|} &= \sum_{A\in\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}} \binom{2p}{|A|} + \sum_{A\in\mathcal{M}_{\mathcal{A}}} \binom{2p}{|A|} \\
&\leq (m_1 - (k - |\mathcal{A}_2|))\binom{2p}{p} + (m - m_1 + |\mathcal{A}_1| + (k - |\mathcal{A}_2|))\binom{2p}{p+1} + (|\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}| - |\mathcal{A}_1|)\binom{2p}{p+2}
\end{aligned}$$

$$\begin{aligned}
&\leq m_1\binom{2p}{p} + (m - m_1 + |\mathcal{A}_1|)\binom{2p}{p+1} + (|\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}| - |\mathcal{A}_1|)\binom{2p}{p+2}\\
&\leq p\binom{2p}{p} + \frac{1}{2}(2m+t-p)\binom{2p}{p+1} + \frac{1}{2}(m-p)\binom{2p}{p+2}\\
&\leq p\binom{2p}{p} + \frac{1}{2}(p+m+d)\binom{2p}{p+1} + \frac{1}{2}(m-p)\binom{2p}{p+2}\\
&\leq p\binom{2p}{p} + \frac{3}{2}p\binom{2p}{p+1} + \frac{1}{2}(p-d)\binom{2p}{p+2}\\
&\leq p\binom{2p}{p} + \frac{3}{2}p\binom{2p}{p+1} + \frac{1}{2}(p-1)\binom{2p}{p+2}.
\end{aligned}$$

The second inequality holds beacuse $\mathcal{A}_2 \leq k$ and the coefficient of $k - |\mathcal{A}_2|$ is $\binom{2p}{p+1} - \binom{2p}{p}$ which is less then 0. The third inequality holds because

- $1 \leq m_1 \leq p, |\mathcal{A}_1| \leq \frac{m_1+t}{2}$;
- By (2.2), $|\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}| \leq \frac{m+t}{2}$;
- The coefficient of m_1 is $\binom{2p}{p} - \binom{2p}{p+1}$ which is larger then 0;
- The coefficient of $|\mathcal{A}_1|$ is $\binom{2p}{p+1} - \binom{2p}{p+2}$ which is larger then 0;

The forth inequality holds beacause of (2.6). The fifth inequality holds because $m \leq n - d$ and the coefficient of m is $\frac{1}{2}\left(\binom{2p}{p+1} - \binom{2p}{p+2}\right)$ which is large than 0. The last inequality holds because $d \geq 1$ and the cofficient of d less than 0.

Since only one of $\frac{3}{2}p$ and $\frac{1}{2}(p-1)$ is an integer, we have $\sum\limits_{A\in\mathcal{A}} \binom{2p}{|A|} \leq p\binom{2p}{p} + \left\lfloor\frac{3}{2}p\right\rfloor\binom{2p}{p+1} + \left\lfloor\frac{1}{2}(p-1)\right\rfloor\binom{2p}{p+2}$. It is easy to see that $p\binom{2p}{p} + \left\lfloor\frac{3}{2}p\right\rfloor\binom{2p}{p+1} + \left\lfloor\frac{1}{2}(p-1)\right\rfloor\binom{2p}{p+2} \geq 2p\binom{2p}{p+1} + p\binom{2p}{p+2}$. □

It is easy to find such $\mathcal{A}$ which reach the upper bound in Theorem 3.3. For example, let $\mathcal{A}$ is union of $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$ where

$$\mathcal{M}_{\mathcal{A}} = \bigcup_{i=1}^{p}\{\pi_i^p\} \cup \bigcup_{i=p+1}^{2p-1}\{\pi_i^{p+1}\},$$

$$\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}} = \begin{cases} \bigcup\limits_{i=1}^{\frac{p}{2}-1}\{\pi_{2i}^{p+1}\} \cup \bigcup\limits_{i=\frac{p}{2}}^{p-2}\{\pi_{2i+1}^{p+2}\} \cup \{\pi_{2p}^{p+1}\} \cup \{\pi_p^{p+1}\} & \text{if } p \text{ is even.}\\ \bigcup\limits_{i=1}^{\frac{p-3}{2}}\{\pi_{2i}^{p+1}\} \cup \bigcup\limits_{i=\frac{p+1}{2}}^{p-1}\{\pi_{2i}^{p+2}\} \cup \{\pi_{2p}^{p+1}\} \cup \{\pi_p^{p+1}\} & \text{if } p \text{ is odd.} \end{cases}$$

So the upper bound of Theorem 3.2 is tight.

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