An extremal problem in the cyclic permutation

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Notations

Let n, k be positive integers and $[n] = \{1, 2, \ldots, n\}$ denote the n-element set. Let $2^{[n]}$ be the power set of $[n]$ and a subset of $2^{[n]}$ is called a *family* of $[n]$. We denote the family of all k-elements subset of $[n]$ by $\binom{[n]}{k}$.

Inclusion-free families

A family $\mathcal{F}$ is called *inclusion-free* if for any $F_1, F_2 \in \mathcal{F}$, $F_1 \subsetneq F_2$. As the first theorem in extremal finite set theory, Sperner determined the upper bound of $|\mathcal{F}|$ for inclusion-free families $\mathcal{F}$.

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Theorem (Sperner theorem)

$max|\mathcal{F}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ where the max is taken over all inclusion-free families.

Intersecting families

A family $\mathcal{F}$ is called *intersecting* if for any $F_1, F_2 \in \mathcal{F}$ the intersection $F_1 \cap F_2 \neq \emptyset$. In 1961, Erdős, Ko and Rado gave the upper bound of $|\mathcal{F}|$ for any intersecting family $\mathcal{F}$.

Theorem (Erdős-Ko-Rado, 1961)

$\max |\mathcal{F}| = 2^{n-1}$, *where the* max *is taken over all intersecting families* $\mathcal{F}$.

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Theorem (Erdős-Ko-Rado Theorem)

Let k *(*$1 \leq k \leq \frac{n}{2}$*) be a fixed integer. Then* $\max|\mathcal{F}| = \binom{n-1}{k-1}$ *over all intersecting families* $\mathcal{F} \subset \binom{[n]}{k}$

Cyclic permutation

A *cyclic permutation* π of the elements of $[n]$ is an ordering of the elements along a cycle. A subset A of $[n]$ is called an *interval* (along π) if its element are consecutive along π. The following statements are well-known.

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Theorem

An inclusion-free family $\mathcal{A}$ of intervals along π has at most n elements. If $\mathcal{A}$ has n elements, then all of its elements have same size.

Theorem

For some $1 \leq k \leq \frac{n}{2}$, if $\mathcal{A}$ is an intersecting family of k-element intervals along π, then $|\mathcal{A}| \leq k$.

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2 Main result

Main result of last sememster

A family $\mathcal{F}$ is called V-free if no triple $F_1, F_2, F_3 \in \mathcal{F}$ such that $F_1 \subset F_2$ and $F_1 \subset F_3$.

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Last sememster, we determine the largest size of a intersecting V-free family $\mathcal{A}$ of intervals along a fixed permutation π.

Main result of last sememster

Theorem

For a intersecting V-free family $\mathcal{A}$ of intervals along a fixed permutation π, if $n \geq 5$, then $|\mathcal{A}| \leq \left\lfloor \frac{3}{2}n \right\rfloor$. In particular, if $|\mathcal{A}| = \left\lfloor \frac{3}{2}n \right\rfloor$, then $|A| > \frac{n}{2}$ for any $A \in \mathcal{A}$.

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For an interval on π, we take the first point in clockwise direction as the starting point. Let π_i^j denote the interval on π with starting point i and size j. Here is one construction of $\mathcal{A}$ with maximum size:

$$\mathcal{A} = \bigcup_{i=1}^{n} \{\pi_i^m\} \cup \bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} \{\pi_{2i}^{m+1}\}$$

where $\frac{n}{2} < m \leq n - 2$.

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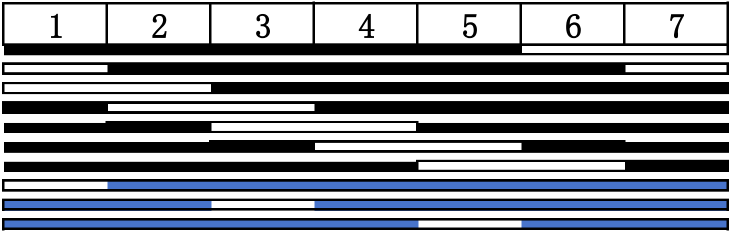
Sum of $|A|$



An example which maximizes $\sum_{A\in\mathcal{A}} |A|$ when $n = 7$.

Sum of $|A|$

Theorem

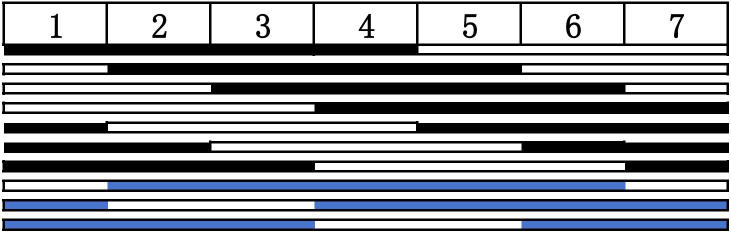
For $n \geq 5$, Assume $\mathcal{A}$ is a V-free intersecting family of intervals on π. Then $\sum\limits_{A \in \mathcal{A}} |A| \leq n(n-2) + \frac{n}{2}(n-1)$.

$$\mathcal{A} = \bigcup_{i=1}^{n} \{\pi_i^{n-2}\} \cup \bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} \{\pi_{2i}^{n-1}\}.$$

is a maximizer of $\sum\limits_{A \in \mathcal{A}} |A|$.

Sum of $\binom{n}{|A|}$, when n is odd

Let $n = 2p + 1$, where $p \geq 2$.



An example which maximizes $\sum_{A \in \mathcal{A}} \binom{2p+1}{|A|}$ when $n = 7$.

Sum of $\binom{n}{|A|}$, when n is odd

Theorem

For $n = 2p + 1, p \geq 2$, assume $\mathcal{A}$ is a V-free intersecting family of intervals on π. Then $\sum_{A \in \mathcal{A}} \binom{2p+1}{|A|} \leq (2p + 1)\binom{2p+1}{p+1} + p\binom{2p+1}{p+2}$

$$\mathcal{A} = \bigcup_{i=1}^{2p+1} \{\pi_i^{p+1}\} \cup \bigcup_{i=1}^{p} \{\pi_{2i}^{p+2}\}$$

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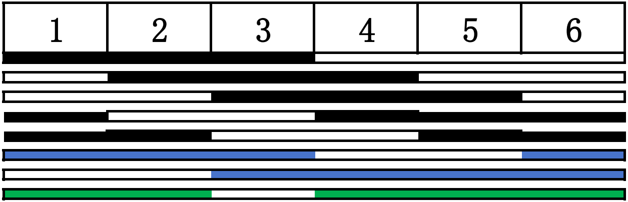
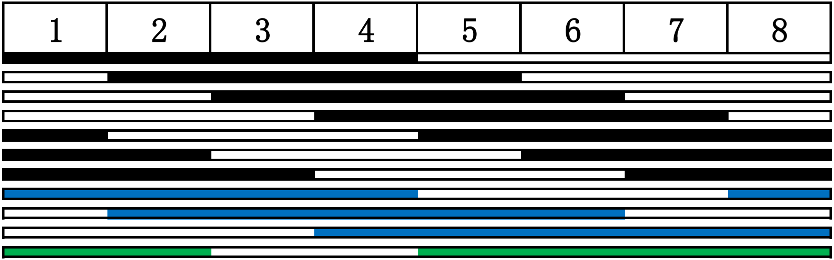
Sum of $\binom{n}{|A|}$, when n is even

Let $n = 2p$, where $p \geq 3$.



An example which maximizes $\sum_{A \in \mathcal{A}} \binom{2p}{|A|}$ when $n = 6$.

Sum of $\binom{n}{|A|}$, when n is even



An example which maximizes $\sum_{A \in \mathcal{A}} \binom{2p}{|A|}$ when $n = 8$.

Sum of $\binom{n}{|A|}$, when n is even

Theorem

For $n = 2p$ with $p \geq 3$, assume $\mathcal{A}$ is a V-free intersecting family of intervals on π. Then $\sum_{A \in \mathcal{A}} \binom{2p}{|A|} \leq p\binom{2p}{p} + \left\lfloor \frac{3}{2}p \right\rfloor \binom{2p}{p+1} + \left\lfloor \frac{1}{2}(p-1) \right\rfloor \binom{2p}{p+2}$.

Sum of $\binom{n}{|A|}$, when n is even

Theorem

For $n = 2p$ with $p \geq 3$, assume $\mathcal{A}$ is a V-free intersecting family of intervals on π. Then $\sum_{A \in \mathcal{A}} \binom{2p}{|A|} \leq p\binom{2p}{p} + \left\lfloor \frac{3}{2}p \right\rfloor \binom{2p}{p+1} + \left\lfloor \frac{1}{2}(p-1) \right\rfloor \binom{2p}{p+2}$.

We can find that:

$$\text{When } n \text{ is even and } \mathcal{A} \text{ maximize} \sum_{A \in \mathcal{A}} \binom{2p}{|A|}, \text{ then } |\mathcal{A}| = \frac{3}{2}n - 1.$$

An example

Let $\mathcal{A}$ is union of $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$ where

$$\mathcal{M}_{\mathcal{A}} = \bigcup_{i=1}^{p}\{\pi_i^p\} \cup \bigcup_{i=p+1}^{2p-1}\{\pi_i^{p+1}\},$$

$$\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}} = \begin{cases} \bigcup\limits_{i=1}^{\frac{p}{2}-1}\{\pi_{2i}^{p+1}\} \cup \bigcup\limits_{i=\frac{p}{2}}^{p-2}\{\pi_{2i+1}^{p+2}\} \cup \{\pi_{2p}^{p+1}\} \cup \{\pi_p^{p+1}\} & \text{if } p \text{ is even.} \\ \bigcup\limits_{i=1}^{\frac{p-3}{2}}\{\pi_{2i}^{p+1}\} \cup \bigcup\limits_{i=\frac{p+1}{2}}^{p-1}\{\pi_{2i}^{p+2}\} \cup \{\pi_{2p}^{p+1}\} \cup \{\pi_p^{p+1}\} & \text{if } p \text{ is odd.} \end{cases}$$

Thank you for listening!