Is there a largest small set?

Kocsis Anett

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The basic question

- Given an ideal $\mathcal{I}$ on X, and a group $G \leq \mathrm{Sym}(X)$, such that $\mathcal{I}$ is invariant under G,
- that is, if $I \in \mathcal{I}$ then $gI \in \mathcal{I}$ for any $g \in G$.

Definition

We say that I^* is a largest element of $\mathcal{I}$ with respect to G, if for any $I \in \mathcal{I}$ there is $g \in G$ such that $gI^* \supseteq I$.

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Examples

- $\mathcal{I} =$ countable sets of $\mathbb{R}$, $G = \mathrm{Homeo}(\mathbb{R})$,
- $\mathcal{I} =$ countable sets of $\mathbb{R}$, $G = \mathrm{Sym}(\mathbb{R})$,
- $\mathcal{I} =$ nowhere dense sets in $[0,1]^d$, $G = \mathrm{Homeo}([0,1]^d)$,
- $\mathcal{I} =$ first category sets in $[0,1]^d$, $G = \mathrm{Homeo}([0,1]^d)$,
- $\mathcal{I} =$ sets of $[0,1]^d$ with Lebesgue measure 0, ~~$G = \mathrm{Homeo}([0,1]^d)$~~, $G = \mathrm{Bilip}([0,1]^d)$,
- $\mathcal{I} =$ compact sets of $\mathbb{N}^\mathbb{N}$, $G = \mathrm{Homeo}(\mathbb{N}^\mathbb{N})$,
- $\mathcal{I} = \sigma$-compact sets of $\mathbb{N}^\mathbb{N}$, $G = \mathrm{Homeo}(\mathbb{N}^\mathbb{N})$,

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Motivation

- recent paper of J. Zapletal
- Permutation models: ZFA models
- The following generalization of largest set turned out to be useful:

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We say that $\mathcal{I}$ has largest element in the strong sense with respect to G, if for any $J \in \mathcal{I}$ there is $I_J \in \mathcal{I}$, such that for any $I \in \mathcal{I}$ there exists $g \in G$ such that $gI_J \supseteq I$, and g fixes J.

- The associated permutation model satisfies the axiom of well-ordered choice iff this holds for the ideal.

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The results

- There is a largest nowhere dense set of $[0,1]^d$, even in the strong sense. The Sierpinski set is a largest set nowhere dense set.
- There is a largest meager set of $[0,1]^d$, but there is no largest in the strong sense.
- There is NO largest zero measure set of $[0,1]^d$.
- There is a largest compact set of $\mathbb{N}^\mathbb{N}$, even in the strong sense. Actually, every perfect compact set is largest.
- There is a largest σ-compact set of $\mathbb{N}^\mathbb{N}$.

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What's next?

- Other applications?
- Other ideals?
- Describing the full hierarchy?

Thank you for your attention!