IS THERE A LARGEST SMALL SET?

KOCSIS ANETT DIRECTED STUDIES II.

1. Introduction

The goal of my directed study was to study largest elements of certain ideals. The motivation for this is the recent paper [3] of J. Zapletal. He proved that in permutation models the well-ordered choice is equivalent to what we call the existence of a largest set in the strong sense. This project is joint work with Márton Elekes and Máté Pálfy. We are planning to publish these result in an upcoming paper.

2. Definition of a largest set

The general setting is the following: let $\mathcal{I} \subseteq \mathcal{P}(X)$ be an ideal, and $G \subseteq \mathrm{Sym}(X)$ a group, such that the ideal is invariant under G, that is, if $I \in \mathcal{I}$ then $gI \in \mathcal{I}$ for any $g \in G$.

Definition 2.1. We say that I^* is *a largest element of $\mathcal{I}$ with respect to G*, if for any $I \in \mathcal{I}$ there is $g \in G$ such that $gI^* \supseteq I$.

The above notion is quite natural, however in [3] the following definition turns out to be usefull.

Definition 2.2. We say that $\mathcal{I}$ has a largest element with respect to G *in the strong sense*, if for any $I \in \mathcal{I}$ there is $I^* \in \mathcal{I}$ such that for any $J \in \mathcal{I}$ there is $g \in G$ such that $gI^* \supseteq J$ and $g(i) = i$ for every $i \in I$.

The goal of this paper is to consider several $(\mathcal{I}, G)$ pair, and examine whether $\mathcal{I}$ has a largest element with respect to G. We examine here the following pairs:

(1) the nowhere dense sets of $[0,1]^d$ with respect to the homeomorphisms of $[0,1]^d$,
(2) the first category sets of $[0,1]^d$ with respect to the homeomorphisms of $[0,1]^d$,
(3) the sets of $[0,1]^d$ with Lebesgue-measure zero with respect to the bilipschitz homeomorphisms of $[0,1]^d$,
(4) the compact sets of $\mathbb{N}^{\mathbb{N}}$ with respect to the homeomorphisms of $\mathbb{N}^{\mathbb{N}}$,
(5) the σ-compact sets of $\mathbb{N}^{\mathbb{N}}$ with respect to the homeomorphisms of $\mathbb{N}^{\mathbb{N}}$.

3. Largest nowhere dense set in $[0,1]^d$

Let $\mathcal{ND}(X)$ denote the set of nowhere dense sets of the topological space X. It is clear that the homeomorphisms of X preserve nowhere density. Thus it is meaningful to ask whether $\mathcal{ND}(X)$ has largest element with respect to $\mathrm{Homeo}(X)$).

We answer this question positively in the case when $X = [0,1]^d$.

Theorem 3.1. *There is a largest element of $\mathcal{ND}([0,1]^d)$ with respect to Homeo$([0,1]^d)$.*

Before proving Theorem 3.1, let us introduce the concept of generalized Sierpinski. Notice that in the two dimensional case our result is closely related to [2].

Intuitively, a generalized Sierpinski set looks like the Sierpinski set, the difference is, that the size of the "holes" may decrease much faster than exponential.

Definition 3.2 (Generalized Sierpinski set)**.** For convenience, we define the complement. First, let us define the finite generations of the complement of a generalized Sierpinski set. Each generation will be the union of disjoint hypercubes and the nth generation will consist of $1 + (3^d - 1) + \dots (3^d - 1)^{n-1}$ many open hypercubes. The first generation is an arbitrary hypercube in $[0,1]^d$, centered at $(0.5, 0.5, \dots 0.5)$. Suppose that the nth generation is already defined. Then the $n+1$th generation is the following: extend the faces of all the hypercubes of the nth generation to hyperplanes. These divide $[0,1]^d$ into $(3^d)^n$ boxes (hyperrectangles), but $(3^d)^{n-1} + (3^d - 1) \cdot (3^d)^{n-2} + \dots + (3^d - 1)^{n-2} \cdot (3^d) + (3^d - 1)n - 1$ from these are the hypercubes of the nth generation. (A hypercube born in the kth generation consist of $(3^d)^{n-k}$ boxes, and in the kth generation $(3^d - 1)^{k-1}$ new hypercubes occour.) So there are $(3^d)^n - (3^d)^{n-1} + (3^d - 1) \cdot (3^d)^{n-2} + \dots + (3^d - 1)^{n-2} \cdot (3^d) + (3^d - 1)^{n-1} = (3^d - 1)^n$ empty boxes. Now pick an ε small enough, then take the hypercubes centered at the center of the empty boxes. Together with the previously defined $1 + (3^d - 1) + \dots (3^d - 1)^{n-1}$ many hypercubes these $1 + (3^d - 1) + \dots (3^d - 1)^n$ hypercubes form the $n+1$th generation.

The generalized Sierpinski set is the complement of the union of all finite generations. It is easy to see that the finite generations together form a dense open set, thus the generalized Sierpinski set is closed nowhere dense.

Notation 3.3. We denote by S_d the basic d dimensional Sierpinski set, that is the generalized Sierpinski set in which the hypercubes of the nth generation have side length $\frac{1}{3^n}$.

Lemma 3.4. *All generalized Sierpinski sets of $[0,1]^d$ are homeomorphic by a homeomorphism of $[0,1]^d$.*

Proof. There is a natural correspondence between the finite generations of the complements. Let the homeomorphism between two corresponding hypercubes be the natural linear map. Since the complements are dense, this extends to a function of $[0,1]^d$, and it is easy to see that it will be a homeomorphism. □

Now we are ready to prove the main theorem of this section.

Proof of Theorem 3.1. We claim that the d dimensional Sierpinski set S_d is a largest set of $\mathcal{ND}([0,1]^d)$ with respect to Homeo$([0,1]^d)$. So fix an arbitrary nowhere dense set N. We are going to create a homeomorphism f and a generalized Sierpinski set S such that $f(S) \supseteq N$. By Lemma 3.4 we know that there is a homeomorphism $g \in \text{Homeo}([0,1]^d)$ such that $g(S_d) = S$. Thus $f \circ g(S_d) \supseteq N$, which proves the maximality of S_d.

Now we create f and S by recursion. We are going to define a series of homeomorphisms $f_0, f_1, \dots f_n \dots$ and open sets $U_0 \subseteq U_1 \subseteq \dots U_n \dots$ such that

$$f_n(x) = f_{n+1}(x) = f_{n+2}(x) = \dots \in [0,1]^d \setminus N \tag{1}$$

for all $n \in \mathbb{N}$ and $x \in U_n$, the uniform limit of $(f_n)_{n\in\mathbb{N}}$ exists and is a homoeomorphism, moreover $\bigcup_{n\in\mathbb{N}} U_n$ is the complement of a generalized Sierpinski set. Finally we define $f := \lim_{n\to\infty} f_n$ and $S := [0,1]^d \setminus \bigcup_{n\in\mathbb{N}} U_n$. From (1) and the fact that $f \in \mathrm{Homeo}([0,1]^d)$ it is clear that $f(S) \supseteq N$.

Let $V \subseteq [0,1]^d \setminus N$ be a dense open set. Let x_0^1 be the center of $[0,1]^d$ and y_0 be a point in V such that $|x_0^1 - y_0| \leq \frac{1}{2}$. Let f_1 be any homeomorphism such that $f_1(x_0^1) = y_0$. Then there is an open hypercube U_0 centered at x_0^1 such that $f_0(U_0) \subseteq V$. Notice that U_0 is the first generation of the complement of a generalized Sierpinski set. Suppose that $f_0, f_1 \dots f_{n-1}$ and $U_0, U_1 \dots U_{n-1}$ are already defined, and U_{n-1} consists of open hypercubes that form the nth generation of the complement of a generalized Sierpinski set. Let $x_n^1, x_n^2, \dots x_n^{(3^d-1)^n}$ be the centers of the $(3^d-1)^n$ many boxes that form the complement of U_n (see Definition 3.2). Since f_{n-1}^{-1} is continuous, there exist δ such that $|x-y| \leq \delta \Rightarrow |f_{n-1}^{-1}(x) - f_{n-1}^{-1}(y)| \leq \frac{1}{2^n}$ for all $x, y \in [0,1]^d$. Let g_n be a homeomorphism with the following properties:

(a) $g_n|_{f_{n-1}(\overline{U_{n-1}})} = \mathrm{id}$,
(b) $g_n(f_{n-1}(x_n^i)) \in V$ for every $i \leq (3^d-1)^n$,
(c) $|g_n - \mathrm{id}\,| \leq \min\{\delta, \frac{1}{2^n}\}$.

The existence of such g_n follows from the fact that V is dense and $x_n^i \notin f_{n-1}(U_{n-1})$ for every $i \leq (3^d-1)^n$.

Set $f_n := g_n \circ f_{n-1}$. Since $f_n(x_n^i) \in V$ for every $i \leq (3^d-1)^n$, there is an ε such $f_n(Q_i) \subseteq V$ $i \leq (3^d-1)^n$, where Q_i that the hypercube of sidelenght ε centered at x_n^i. Let $U_n := \bigcup_{i\leq(3^d-1)^n} Q_i$.

Now we prove that the defined $f_0, f_1 \dots f_n \dots$s and $U_0, U_1 \dots U_n \dots$ satisfy the required properties. It is clear that $\bigcup_{n\in\mathbb{N}} U_n$ is the complement of a generalized Sierpinski set, since $\bigcup_{k\leq n} U_k$ is the $n+1$th generation of the complement of a generalized Sierpinski set. Moreover (1) follows from (a). For the existence of the uniform limit $(f_n)_{n\in\mathbb{N}}$ we have to check that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. This is so, since from (c) we have:

$$|f_n(x) - f_{n-1}(x)| = |g_n(f_{n-1}(x)) - f_{n-1}(x)| \leq \frac{1}{2^n}$$

for all $x \in [0,1]^d$. On the other hand $(f_n^{-1})_{n\in\mathbb{N}}$ is also a Cauchy sequence, since $|g_n^{-1}(x) - x|\delta$ and thus we have:

$$|f_n^{-1}(x) - f_{n-1}^{-1}(x)| = |f_{n-1}^{-1}(g_n^{-1}(x)) - f_{n-1}^{-1}(x)| \leq \frac{1}{2^n}.$$

It is easy to check that the uniform limit of $(f_n^{-1})_{n\in\mathbb{N}}$ is the inverse of the uniform limit of $(f_n)_{n\in\mathbb{N}}$. So f is indeed a homeomorphism, which finishes the proof. □

For the sake of completeness let we remark that the following strengthening of Theorem 3.1 is also true.

Theorem 3.5. *There is a largest nowhere dense set with respect to* $\mathrm{Homeo}([0,1]^d)$ *in the strong sense.*

We also have the following theorem, but we omit the proof.

Theorem 3.6. *There is a largest first category set in* $[0,1]^d$ *with respect to* $\mathrm{Homeo}([0,1]^d)$.

4. Largest compact set in the Baire space

In this section we discuss the case when $G = \mathrm{Homeo}(\mathbb{N}^{\mathbb{N}})$ and $\mathcal{I}$ is the ideal of compact sets in $\mathbb{N}^{\mathbb{N}}$, that is $\mathcal{I} = \mathcal{K}(\mathbb{N}^{\mathbb{N}})$. It turns out, that in this case all non-countable candidates are largest:

Theorem 4.1. *Let C be a compact set in the Baire space that is not countable. Then C is a largest set of $\mathcal{K}(\mathbb{N}^{\mathbb{N}})$ with respect to* $\mathrm{Homeo}(\mathbb{N}^{\mathbb{N}})$.

Proof. Let us fix $C_0 \in \mathcal{K}(\mathbb{N}^{\mathbb{N}})$. We are going to construct a homeomorphism f such that $f(C) \supseteq C_0$. Let $C_1 = \prod_{n\in\mathbb{N}} \pi_n(C_0) \subseteq \mathbb{N}^{\mathbb{N}}$, then it sufficies to find f for which $f(C) \supseteq C_1$. For later convenience let a_n denote the possible nth coordinates of C_1, that is $a_n := |\pi_n(C_1)|$. For any $s \in \mathbb{N}^{<\mathbb{N}}$ let $\mathcal{N}_s$ denote the usual basic clopen set, that is $\mathcal{N}_s = \{x \in \mathbb{N}^{\mathbb{N}} : x|_{|s|} = s\}$. By throwing away a countable set we may suppose that C is perfect. Let us introduce the following notation. For a compact set $K \subseteq \mathbb{N}^{\mathbb{N}}$

$$K|_n := \{s \in \mathbb{N}^n : \exists x \in K,\ x|_n = s\}.$$

We will define a strictly monotone increasing sequence the following way. Let n_0 be the least postive integer such that $|C|_{n_0}| \geq a_0$. Suppose that n_k is already defined, then let n_{k+1} be the smallest natural number for which $n_{k+1} > n_k$ and $|(C \cap \mathcal{N}_s)|_{n_{k+1}}| \geq a_{k+1}$ for every $s \in C|_{n_k}$. Since C is perfect, it is clear that such n_{k+1} exists.

Now we construct the homeomorphism f. First, we will construct recursively for all $k \in \mathbb{N}$ a bijection $e_k : \mathbb{N}^{n_k} \to \mathbb{N}^k$ such that

$$e_k(C|_{n_k}) \supseteq \prod_{n=1}^{k} \pi_n(C_1) \tag{2}$$

and these functions extend each other. That is, for any k and $s \in \mathbb{N}^{n_{k+1}}$

$$e_k(s|_{n_k}) = e_{k+1}(s)|_k. \tag{3}$$

Suppose that e_k is already defined. First we want to define e_{k+1} on $C|_{n_{k+1}}$. There are a_{k+1} many extensions for every $s \in \prod_{n=1}^{k} \pi_n(C_1)$. Recall that $|(C \cap \mathcal{N}_t)|_{n_{k+1}}| \geq a_{k+1}$ for every $t \in C|_{n_k}$, that is, there exists $s_1, \dots s_{a_{k+1}} \in \mathbb{N}^{n_{k+1}-n_k}$ such that $t^\frown s_i \in C|_{n_{k+1}}$. So for every $t \in C|_{n_k}$ for which $e_k(t) \in \prod_{n=1}^{k} \pi_n(C_1)$ let us define

$$e_{k+1}(t^\frown s_i) := e_k(t)^\frown u_i,$$

where $\{u_1, u_2 \dots u_{a_{k+1}}\} = \pi_n(C_1)$. This construction ensures that e_{k+1} satisfies (1). As we have defined e_{k+1} on finitely many elements of $\mathbb{N}^{n_{k+1}}$, and every $s \in \mathbb{N}^{n_k}$ and every $t \in \mathbb{N}^k$ can be extended infinitely many ways, it is easy to construct a bijection that satisfies (2).

Let $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the unique common extension of the e_k's, that is $f(x) = y \iff \forall k \; e_k(x|_{n_k}) = y|_k$. It is easy to see that f is a well defined bijection. It is a homeomorphism, since by (2) $f(\mathcal{N}_s) = \mathcal{N}_{e_k(s)}$ for any $s \in \mathbb{N}^{n_k}$, and $f^{-1}(\mathcal{N}_t) = \mathcal{N}_{e_k^{-1}(t)}$ for any $t \in \mathbb{N}^k$. Finally from (1) it is clear that $f(C) \supseteq C_1$, which finishes the proof. □

As an easy corollary of Theorem 4.1 we get the following

Corollary 4.2. *There is a largest set in* $\mathcal{K}(\mathbb{N}^{\mathbb{N}})$ *with respect to* $\mathrm{Homeo}(\mathbb{N}^{\mathbb{N}})$ *in the strong sense.*

Proof. Let C^* be the compact set that has to be fixed. We claim that any C perfect compact set for which $C^* \cap C = \emptyset$ is C^*-largest. To show this, let $C_1 \in \mathcal{K}(\mathbb{N}^{\mathbb{N}})$ such that $C_1 \cap C^* = \emptyset$. Then there exists a clopen partition $U \sqcup V = \mathbb{N}^{\mathbb{N}}$ such that $C^* \subseteq U$ and $C_1 \cup C \subseteq V$. It is well known that any clopen subset of $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. Thus from Theorem 4.1 we know that there is $f_0 : V \to V$ for which $f(C) \supseteq C_1$. Now let us define f as follows:

$$f(x) = \begin{cases} f_0(x), \text{ if } x \in U \\ x, \text{ if } x \in V. \end{cases}$$

Clearly f is a homeomorphism, as it is the union of homeomorphisms on disjoint clopen sets. It is also clear from definition, that $f(C) \supseteq C_1$ and $f|_{C^*} = \mathrm{id}$, and that is what we wanted to show. □

We also have the following theorem, but we omit the proof.

Theorem 4.3. *There is a largest σ-compact set in the Baire space with respect to* $\mathrm{Homeo}([\mathbb{N}^{\mathbb{N}})$.

5. There is no largest zero measure set of $[0,1]^d$

We also would like to examine the family of zero-measure sets. Since we want the group G to preserve our ideal, it is natural to choose G to be the group of bilipschitz bijections of the $[0,1]^d$ cube.

Let us denote with $\mathcal{N}([0,1]^d)$ the family of sets with measure zero in the $[0,1]^d$ cube, and $\mathrm{Bilip}([0,1]^d)$ the family of bilipschitz bijections of $[0,1]^d$.

Theorem 5.1. *There is no largest element of* $(\mathcal{N}([0,1]^d))$ *with respect to Bilip*$([0,1]^d)$.

For the proof of 5.1 the key proposition is Proposition 5.3. Let us introduce the following notation:

Notation 5.2. Let $\mathcal{K}^{\frac{1}{2}+}\left([0,1]^d\right)$ denote the set of compact sets of $[0,1]^d$, which have Lebesgue-measure at least $\frac{1}{2}$.

It is easy to see that $\mathcal{K}^{\frac{1}{2}+}\left([0,1]^d\right)$ is closed subset of $\mathcal{K}\left([0,1]^d\right)$ (the space of compact sets in $[0,1]^d$), and thus it forms a Polish space.

Proposition 5.3. *Let C be a Cantor set of positive measure in $[0,1]^d$. Then there is a residual set $R \subseteq \mathcal{K}^{\frac{1}{2}+}\left([0,1]^d\right)$, such that for any $C' \in R$ there is no $f \in Bilip([0,1]^d)$, for which $f(C) \subseteq C'$.*

Before we start to prove Proposition 5.3, let us present the following easy, but useful statements:

Lemma 5.4. *Let $C \subseteq [0,1]$ be a Cantor set with $\lambda(C) > 0$. Then there is a srtictly monotone decreasing sequence a_n with limit point a, such that $a_n \in C$ for all $n \in \mathbb{N}$, furthermore $\lim_{n\to\infty} \frac{a_{n+1}-a}{a_n-a} = 1$.*

Proof. We know from the Lebesgue density theorem, that almost every point of C is a density point, so let a be such. We are going to construct a_n by recursion. Let us choose furthermore $b_1, b_2 \dots b_n \dots \geq a$ such that $\frac{\lambda([a,x]\cap C)}{[a,x]} > \frac{1}{4(i+1)}$ for all $a < x \leq b_i$. They exist because of the assumption that a is a density point. Of course, we can suppose that $(b_n)_{n\in\mathbb{N}}$ is strictly monotone decreasing and its limit is a. Let $a_1 = b_1$. Suppose now, that a_k is already defined and i is the smallest integer such that $b_i < a_k$. Since $a_k \leq b_{i-1}$, we know that $\frac{\lambda([a,a_k]\cap C)}{[a,a_k]} > \frac{1}{4i}$, therefore $C \cap [a + (a_k - a)\frac{4i-2}{4i}, a + (a_k - a)\frac{4i-1}{4i}] \neq \emptyset$. Then let a_{k+1} be an arbitrary element of $C \cap [a + (a_k - a)\frac{4i-2}{4i}, a + (a_k - a)\frac{4i-1}{4i}]$. Notice that a_n is monotone decreasing, and $a_n > a$ for all $n \in \mathbb{N}$. We show that $a_n \to a$. It is enough to see that for any i there is k such that $a_k < b_i$. But this is so, since $a_{l+1} - a < \frac{4i-1}{4i}(a_l - a)$ for all $a_l > b_i$, therefore there are only finitely many a_l's greater than b_i.

So we only have to check that $\lim_{n\to\infty} \frac{a_{n+1}-a}{a_n-a} = 1$. Fix i, then for all k such that $a_k \leq b_i$ we know that $a_k - a \geq a_{k+1} - a \geq (a_k - a)\frac{4(i+1)-2}{4(i+1)}$, which finishes the proof. □

The next statement is the key lemma for Proposition 5.3. The statement and the proof are essentially the same as Theorem 1.2 in [1], we present it here only for the sake of completeness. The only difference is that we prove the statement for higher dimension and for arbitrary tail $(a_n)_{n=n_0}^{\infty}$.

Lemma 5.5. *Let $(a_n)_{n=1}^{\infty} \subseteq [0,1]$ be a strictly decreasing sequence with $a_n \to a$ as $n \to \infty$. If*

$$\lim_{n\to\infty} \frac{a_{n+1}-a}{a_n-a} = 1,$$

then there exists a compact set $E \subset [0,1]^d$ with positive Lebesgue measure such that for all bilipschitz function $f : [0,1] \to [0,1]^d$ there are infinitely many indices k such that $f(a_k) \notin E$.

Proof. It is enough to prove the statement for a subsequence, so applying Lemma 3.1 in [1] to the sequence $(a_n - a)_{n\in\mathbb{N}}$ we may suppose that

$$a_n - a_{n+1} \leq 2(a_m - a_{m+1}) \quad \text{for all } n, m \text{ with } n > m. \tag{4}$$

Since

$$\lim_{n\to\infty} \frac{a_n - a_{n+1}}{a_n - a} = 1 - \lim_{n\to\infty} \frac{a_{n+1}-a}{a_n-a} = 0,$$

we can choose a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of natural numbers such that

$$\frac{a_{n_k} - a_{n_k+1}}{a_{n_k} - a} \leq \frac{1}{d} k^{-2} 4^{-k} \quad \text{for } k \geq 1.$$

For each $k \geq 2$, let ℓ_k be the smallest integer $\geq k/(a_{n_k} - a)$, and let $\delta_k = k(a_{n_k} - a_{n_k+1})$. Clearly, it holds that

$$\frac{1}{\ell_k} \leq \frac{a_{n_k} - a}{k} < \frac{2}{\ell_k}$$

and

$$\ell_k \delta_k \leq \frac{2k}{a_{n_k} - a} \cdot k(a_{n_k} - a_{n_k+1}) \leq \frac{1}{d} \cdot 2 \cdot 4^{-k} \quad \text{for } k \geq 2. \tag{5}$$

Define a sequence $(E_k)_{k=1}^{\infty}$ of compact subsets of $[0,1]^d$ by

$$E_k = [0,1]^d \setminus \bigcup_{i=1}^{d} \bigcup_{j=0}^{\ell_k} \pi_i^{-1} \left[\left(\frac{j}{\ell_k} - \frac{\delta_k}{2}, \frac{j}{\ell_k} + \frac{\delta_k}{2} \right) \right],$$

where π_i is the projection to the ith coordinate. It is easy to see that for each k, E_k is the union of $(\ell_k)^d$ disjoint cubes of sidelength $(1 - \delta_k \ell_k)/\ell_k$, with a distance of length δ_k between any two adjacent cubes.

Set $E = \bigcap_{k=1}^{\infty} E_k$. Then E is a compact set with Lebesgue measure

$$\lambda(E) \geq 1 - \sum_{k=1}^{\infty} \lambda\left([0,1]^d \setminus E_k]\right) \geq 1 - \sum_{k=1}^{\infty} d\ell_k \delta_k \geq 1 - \sum_{k=1}^{\infty} 2 \cdot 4^{-k} = \frac{1}{3} > 0,$$

where we have used (5) in the third inequality. Now we are ready to show that for any bilipschitz map $f : [0,1] \to [0,1]^d$ infinitely many elements of $(a_n)_n \in \mathbb{N}$ are not mapped into E.

Suppose on the contrary that there exists n_0 such that $(a_n)_{n=n_0}^{\infty}$ can be embedded into E by a bilipschitz map $f : [0,1] \to [0,1]^d$. Let $b_n = f(a_n)$ for $n \geq n_0$ and $b_\infty = \lim_{n\to\infty} b_n$. Then $b_n, \ b_\infty \in E$. Clearly $b_\infty = f(a)$. Since f is bilipschitz, there exists a constant $L > 1$ such that

$$L^{-1} \leq \frac{|b_n - b_m|}{a_n - a_m} \leq L \quad \text{for all } n_0 \leq n < m.$$

In particular, this implies that

$$L^{-1} \leq \frac{|b_n - b_{n+1}|}{a_n - a_{n+1}} \leq L \quad \text{and} \quad L^{-1} \leq \frac{|b_n - b_\infty|}{a_n - a} \leq L. \tag{6}$$

Now fix an integer $k > \max\{2\sqrt{d}L, n_0\}$. Then by (6) and (4), for all $m \geq n_k$,

$$|b_m - b_{m+1}| \leq L(a_m - a_{m+1}) \leq 2L(a_{n_k} - a_{n_k+1}) < k(a_{n_k} - a_{n_k+1}) = \delta_k. \tag{7}$$

Meanwhile by (6),

$$|b_{n_k} - b_\infty| \geq \frac{a_{n_k} - a}{L} > \sqrt{d}\frac{a_{n_k} - a}{k} \geq \frac{\sqrt{d}}{\ell_k}. \tag{8}$$

Notice that $(b_m)_{m=n_k}^{\infty} \subset E \subset E_k$. Recall that E_k is the union of $(\ell_k)^d$ disjoint cubes of sidelength $(1 - \delta_k \ell_k)/\ell_k$, with a distance δ_k between any two adjacent intervals. By (7), the sequence $(b_m)_{m=n_k}^{\infty}$ must be entirely contained in a cube of E_k. This forces that $|b_{n_k} - b_\infty| \leq \sqrt{d}(1 - \delta_k \ell_k)/\ell_k$, which clearly contradicts (8). $\square$

We are ready to prove the following:

Proof of Proposition 5.3. We are going to show that the set of those compact sets (with Lebesgue-measure at least $\frac{1}{2}$) in $[0,1]^d$, in which C is not embeddable by a bilipschitz function, is dense G_δ. By the Baire category theorem and the fact that $\mathcal{K}^{\frac{1}{2}+}\left([0,1]^d\right)$ is Polish, this shows that these sets form a residual subset in $\mathcal{K}^{\frac{1}{2}+}\left([0,1]^d\right)$. First, let us show that it is G_δ. For this it suffices to show that for fixed $K \in \mathbb{N}$ the following set is closed:

$$\mathcal{E}_K := \{X \in \mathcal{K}^{\frac{1}{2}+}\left([0,1]^d\right) : \exists f \in \text{Bilip}([0,1]^d) \text{ with bilipschitz constant } K \text{ and } f(C) \subseteq X\}.$$

Suppose that X_n is a convergent subsequence of $\mathcal{E}_K$ with limit X. Let us fix for every X_n a witness bilipschitz function f_n with bilipschitz constant K. By the Arzela-Ascoli theorem we can take a subsequence $(n_k)_{k\in\mathbb{N}}$ such that there exists a function f for which $f_{n_k} \to f$ in supremum norm. Then of course $f \in \text{Bilip}([0,1]^d)$ with bilipshitz constant K. For any $x \in C$

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) \text{ and } f_{n_k}(x) \in X_{n_k},$$

thus $f(x) \in X$, which proves that $\mathcal{E}_K$ is closed.

For the denseness, fix an arbitrary compact set C' with (d-dimensional) Lebesgue measure at least $\frac{1}{2}$ and $\varepsilon > 0$. By Fubini theorem there exists $(c_1, c_2 \dots c_{d-1}) \in [0,1]^{d-1}$ such that $C_{c_1,c_2\dots c_{d-1}}$ has positive (1-dimensional) Lebesgue measure. Thus we know from Lemma 5.4 and Lemma 5.5 that there is a strictly monotone sequence $(a_n)_{n\in\mathbb{N}} \subseteq C_{c_1,c_2\dots c_{d-1}}$ and a compact set E with positive Lebesgue measure such that for every k, $(a_n)_{n=k}^{\infty}$ is not embeddable with a bilipschitz function into E. From a basic exhaustion argument we know that there are $E_1, E_2 \dots E_m$ finitely many disjoint affine copies of E such that $d_H(\bigcup_{i=1}^m E_i, C') < \varepsilon$ and $\bigcup_{i=1}^m E_i$ has Lebesgue measure at least $\frac{1}{2}$. We claim that C is not embeddable into $\bigcup_{i=1}^m E_i$ with a bilipschitz bijection, for which is suffices to show that $(a_n)_{n\in\mathbb{N}}$ is not embeddable with the same type of function. But this holds, since suppose on the contrary that $f(\{(a_n)_{n\in\mathbb{N}}\}) \subseteq \bigcup_{i=1}^m E_i$. Then $f(a) \in E_i$ for some $1 \le i \le m$. But all the E_j's are disjoint, so they have positive distance, thus from the continuity of f all but finitely many elements of $f(a_n)_{n\in\mathbb{N}}$ are in E_i. This gives a contradiction with Lemma 5.5, and thus the proof is finished.

□

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. Suppose that there is a largest element of $(\mathcal{N}([0,1]^d), \text{Bilip}[0,1]^d)$, let us call it G. By taking a hull we can suppose that G is a G_δ set. Let us choose a Cantor set C with the property, that $\lambda_d(C) \ge \frac{1}{2}$, $C \cap G = \emptyset$ and for every open ball $B_\varepsilon(x) \subseteq [0,1]^d$ if $B_\varepsilon(x) \cap C \ne \emptyset$ then $\lambda(B_\varepsilon(x) \cap C) > 0$. [TODO: indokoljuk, hogy miért lehet ilyen C-t venni?]

Let $B_1, B_2 \dots$ be an enumeration of those open balls, which have rational radius, their center has rational coordinates and $B_i \cap C \ne \emptyset$. We may apply Proposition 5.3 and the Baire category theorem and conclude that the following set

$$\{C' \in \mathcal{K}^{\frac{1}{2}+}\left([0,1]^d\right) : \forall i \; C \cap B_i \text{ is not embeddable into } C' \text{ with a bilipschitz bijection}\}$$

is comeager. Let us choose such a set C'. It is easy to check that $F' := (C' + \mathbb{Q}^d) \cap [0,1]^d$ is a set of full measure. We claim that the complement of F' proves that G is not a largest element of $(\mathcal{N}([0,1]^d), \mathrm{Bilip}[0,1]^d)$. It is enough to show the following:

Claim: There is no bilipschitz bijection f, such that $f(G) \supseteq [0,1]^d \setminus F'$.

Proof of the claim: It is equivalent to show that there is no bilipschitz bijection f such that $f([0,1]^d \setminus G) \subseteq F'$. Suppose towards a contradiction that there is such a function, let us call it f. Let $q_0, q_1, q_2 \ldots$ be an enumeration of $\mathbb{Q}^d$. Then $F' = \bigcup_{i \in \mathbb{N}} (q_i + C') \cap [0,1]^d$, thus according to our assumption $f(C) \subseteq f([0,1]^d \setminus G) \subseteq \bigcup_{i=1}^{\infty} (q_i + C') \cap [0,1]^d$. Therefore:

$$C \subseteq \bigcup_{i=1}^{\infty} f^{-1}((q_i + C') \cap [0,1]^d)$$

It is clear from the Baire category theorem, that for some i the set $f^{-1}((q_i + C') \cap [0,1]^d)$ has nonempty interior in C. This equivalent to say that there is a ball B_j such that $B_j \cap C \subseteq f^{-1}((q_i + C') \cap [0,1]^d)$. In other words:

$$f(B_j \cap C) \subseteq (q_i + C') \cap [0,1]^d.$$

But we can suppose that $\operatorname{diam} f(B_j \cap C)$ is small enough and thus there is a a bilipschitz bijection g of $[0,1]^d$ for which $g(f(B_j \cap C)) = f(B_j \cap C) - q_i \subseteq C'$. Thus $B_j \cap C$ is bilipschitz embeddable into C' via $g \circ f$, contradicting to our choice of C'.

□

6. Open questions

We conclude this report with some open questions:

Question 6.1. It is interesting to examine not just the largest elements of the poset but also the structure of the hierarchy. For example in the case of compact sets of the Baire space we have proved that a set is largest (with respect to $\mathrm{Homeo}(\mathbb{N}^{\mathbb{N}})$) if and only if it is not countable. However, in the other cases the situation might be more complicated.

Question 6.2. There are many more natural ideals for which we can examine the existence of a largest set, for example the ideal of the Haar null or Haar meager sets. In this case, the natural group would be the topological group automorphisms.

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