On some conjectures about basis exchanges and orderings

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Continuing the research from last semester, I started with investigating the conjecture of Kajitani et. al. [5] about cyclically orderable matroids.

Definition 1. A matroid M of rank r is **cyclically orderable** if there exists a cyclic permutation of the elements of M such that any r consecutive elements form a base.

Definition 2. A matroid M with ground set S is called **uniformly dense** if $r(S) \cdot |X| \leq r(X) \cdot |S|$ holds for every $X \subseteq S$.

Conjecture 1 (Kajitani, Ueno, Miyano)**.** *A matroid M is cyclically orderable if and only if it is uniformly dense.*

Van den Heuvel and Thomassé [6] showed that the conjecture is true if $|S|$ and $r(S)$ are coprimes.

Theorem 3 (van den Heuvel, Thomassé)**.** *Let M be a loopless matroid such that $|S|$ and $r(S)$ are coprime. There is a cyclic ordering of S such that every $r(S)$ cyclically consecutive elements are bases of M if and only if M is uniformly dense.*

Throughout the proof, they use a technical theorem about weighted matroids. A weighted matroid is a matroid with a weight function $w : S \to \mathbb{R}$ on the ground set.

Theorem 4 (van den Heuvel, Thomassé)**.** *Let (M, w) be a loopless weighted matroid with non-negative integer weights and D a positive integer. Then there* *exists a mapping $\varphi : S \to \{1, 2, \ldots, D\}$ such that for every $x \in \{1, 2, \ldots, D\}$, the set $\{s \in S \mid x \in [\varphi(s), \varphi(s) + w(s))\}$ is independent if and only if for all $X \subseteq S$, we have $w(X) \leq D \cdot r(X)$.*

Proof of Theorem 3. If there exists a cyclic ordering of S, then any element of the ground set is contained in r of the bases formed by r consecutive elements. Let us denote the aforementioned bases by $B_1, \ldots, B_n$, where $|S| = n$. So for any $X \subseteq S$, $r \cdot |X| = |X \cap B_1| + \ldots + |X \cap B_n| \leq n \cdot r(X)$.

For the other direction, we use Theorem 4 with $w(s) = r$ for all $s \in S$ and $D = n$. The condition is satisfied by the assumption of uniform density. So there exists a mapping $\varphi : S \to \{1, 2, \ldots, n\}$ such that for every $x \in \{1, 2, \ldots, n\}$, the set $S_x = \{s \in S \mid x \in \{\varphi(s), \varphi(s) + 1, \ldots, \varphi(s) + r - 1\}\}$ is independent. So since all weights are r, the elements mapped to the cyclic interval $I_r(x) = \{x, x + 1, \ldots, x + r - 1\}$ form an independent set. By double counting, each interval needs to have r elements mapped to it, because by independence, there cannot be more than r mapped to any of the intervals.

Suppose that there is an $x \in \{1, 2, \ldots, n\}$ without a preimage :((((. Then there are r elements mapped to $\{x + 1, \ldots, x + r - 1\}$, so $x + r$ has no preimage either. By repeating this argument, $x + 2r, x + 3r, \ldots$ have no preimage, and by the assumption of $gcd(n, r) = 1$, this process reaches all elements, which is a contradiction. So every x has a preimage, which means they all have exactly one, and that gives a cyclic ordering given that any r consecutive elements form an independent set.

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Throughout the proof, we only use the assumption of $gcd(n, r) = 1$ once, in the end. If $gcd(n, r) = g$, we can repeat the same proof, except that in the last paragraph we look at the preimage of intervals of length g. If one of them, say $\{x, x+1, \ldots, x+g-1\}$, has a preimage of size less than g, then $\{x+g, \ldots, x+r-1\}$ has a preimage of size more than $r-g$, so then $\{x+r, \ldots, x+r+g-1\}$ has also a preimage of size less than g. By repeating this argument, we get n/g disjoint intervals of length g which all have a preimage of size smaller than g, which is a contradiction. This means that all intervals of length g have a preimage of size g. This reduces the original problem in the following way.

Question 5. *Let M be a matroid with ground set S, $|S| = n$ and rank r, and $gcd(n,r) = g$. If the ground set is partitioned into n/g groups with size g along a circle and any r/g cyclically consecutive groups form a basis, then is there an ordering within the groups such that any r consecutive elements form a basis?*

If the answer was yes, then it would solve the original conjecture for any matroid, because we always have such a partitioning by the modified proof.

However, the answer is no, we found multiple counterexamples. At first, we did not assume that g is the greatest common divisor, just any common divisor. For that, sparse paving matroids give a counterexample with $n = 8$, $r = 4$, $g = 2$. For that, we used the hypergraph representation of sparse paving matroids, given by Bérczi et. al. [2]. Let the groups be $\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}, \{a_7, a_8\}$. Let the hyperedges be $\{a_1, a_2, a_3, a_8\}, \{a_1, a_2, a_4, a_7\}, \{a_3, a_5, a_6, a_7\}, \{a_4, a_5, a_6, a_8\}$. Those all have intersection of size at most 2, so they represent a sparse paving matroid. Then the ordering of $\{a_1, a_2\}$ and $\{a_5, a_6\}$ does not matter, but all the 4 ways in which $\{a_3, a_4\}$ and $\{a_7, a_8\}$ could be ordered, are excluded by one of the hyperedges.

With the assumption of $g = gcd(n, r)$, there is also a counterexample with sparse paving matroids. Let $n = 10, r = 4, g = 2$. Let the groups be $\{a_1, a_2\}, \{a_3, a_4\}$, $\{a_5, a_6\}, \{a_7, a_8\}, \{a_9, a_{10}\}$. Let the hyperedges be $\{a_1, a_2, a_3, a_{10}\}, \{a_1, a_2, a_4, a_9\}$, $\{a_1, a_3, a_4, a_5\}, \{a_2, a_3, a_4, a_6\}, \{a_3, a_5, a_6, a_7\}, \{a_4, a_5, a_6, a_8\}, \{a_5, a_7, a_8, a_9\}$, $\{a_6, a_7, a_8, a_{10}\}, \{a_1, a_7, a_9, a_{10}\}, \{a_2, a_8, a_9, a_{10}\}$. Those all have intersection of size at most 2, so they represent a sparse paving matroid. The ordering cannot be done because all the pairs have 2 hyperedges containing them, which excludes 2 of the 4 possibilities of their neighbouring groups, but at the end of the circle, there is no remaining possibility.

Note that in all of our counterexamples, $g = 2$. In this case, there are not that many possible orderings, which might be the problem. So it still remains an open problem for $g > 2$. There might be some counting arguments that solve the question for large g, but I did not manage to work them out.

There are many related conjectures to cyclic orderings, we mention some of those in the following.

Definition 6. Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be basis pairs of the same matroid. We call $\mathbf{P}_1$ and $\mathbf{P}_2$ **compatible** if the multiunion of R_1 and B_1 is the same as the multiunion of R_2 and B_2.

White's conjecture [7] gives an equivalent condition for compatibility.

Conjecture 2 (White)**.** *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be compatible basis pairs of the same matroid. Then there exists a sequence of symmetric exchanges that transforms the basis pair $\mathbf{P}_1$ into $\mathbf{P}_2$.*

Hamidoune's conjecture [3] makes White's conjecture stronger by giving an upper bound on the number of exchanges.

Conjecture 3 (Hamidoune)**.** *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be compatible basis pairs of a rank-r matroid M over a ground set S. Then the exchange distance of $\mathbf{P}_1$ and $\mathbf{P}_2$ is at most r.*

A weighted version of Hamidoune's conjecture is proposed by Bérczi, Mátravölgyi and Schwarz [1], who also proved it for strongly base orderable matroids, split matroids, graphic matroids of wheels, and spikes.

Conjecture 4 (weighted Hamidoune)**.** *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be compatible basis pairs of a matroid M over a ground set S, and let $w \colon S \to \mathbb{R}_+$ be a weight function. Then the weighted exchange distance of $\mathbf{P}_1$ and $\mathbf{P}_2$ is at most $w(R_1 \cup B_1) = w(R_2 \cup B_2)$.*

By setting the weight function to be identically one, we get back Hamidoune's conjecture. It is worth mentioning that a strictly monotone exchange sequence transforming $\mathbf{P}_1$ into $\mathbf{P}_2$ is optimal in every sense, i.e., it has both minimum length and minimum weight.

It is not clear how to generalize the previous conjectures for multiple bases instead of pairs, but these are some possibilities.

White's conjecture can be generalized with symmetric exchanges between basis sequences.

Conjecture 5 (generalized White)**.** *Let $X = (X_1, X_2, \ldots, X_k)$ and $Y = (Y_1, Y_2, \ldots, Y_k)$ two tuples of k bases with the same multiunion. Then there exists a sequence of symmetric exchanges that transforms the basis tuple X into Y.*

Our proposition is through cyclic exchanges. A cyclic exchange means that in each basis at most one element changes to another element from a different basis, in a way that they all remain bases.

Conjecture 6 (generalized Hamidoune)**.** *Let $X = (X_1, X_2, \dots, X_k)$ and $Y = (Y_1, Y_2, \dots, Y_k)$ two tuples of k bases of a matroid M with the same multiunion. Then there exists a sequence of at most r cyclic exchanges that transforms the basis tuple X into Y.*

This can also be extended to the weighted case. Let the weight of a cyclic exchange be the average of the weight of the elements that are moved in the exchange.

Conjecture 7 (generalized weighted Hamidoune)**.** *Let $X = (X_1, X_2, \dots, X_k)$ and $Y = (Y_1, Y_2, \dots, Y_k)$ two tuples of k bases of a matroid M with the same multiunion, and let $w\colon S \to \mathbb{R}_+$ be a weight function. Then there exists a sequence of cyclic exchanges with total weight at most $w(S)/k$ that transforms the basis tuple X into Y.*

This generalization is sensible because it implies Gabow's conjecture (and its generalization) and Hamidoune's conjecture as well. However, the generalized White's conjecture does not follow from that, because it is about symmetric exchanges (which is only the same as cyclic exchanges in the $k = 2$ case).

Gabow's conjecture [4] is the $k = 2, w \equiv 1$ and $(Y_1, Y_2) = (X_2, X_1)$ case of our new conjecture.

Conjecture 8 (Gabow)**.** *Let B and B' be disjoint bases of the same matroid with rank r. Then there exists a sequence of r symmetric exchanges that transforms the basis pair (B, B') into (B', B).*

The generalized case of Gabow's conjecture (which I proposed in last semester) is the $w \equiv 1$ and $(Y_1, Y_2, \dots, Y_k) = (X_k, X_1, \dots, X_{k-1})$ case of our new conjecture.

Conjecture 9 (generalized Gabow)**.** *Let $B_1, B_2, \dots B_k$ be disjoint bases of the same matroid of rank r. Then there exists a permutation of the elements of $B_1 \cup B_2 \cup \dots \cup B_k$ in which any r cyclically consecutive elements form a basis, and the elements of B_i are r cyclically consecutive elements for all $1 \leq i \leq k$.*

The further goals are to write an article about our results and investigate special cases of the new, most general conjecture.

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