Forcing and failure of GCH*

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1 Introduction

In modern set theory, forcing and large cardinals represent two central topics. In the following we introduce a standard forcing technique, called iterated forcing, and employ it to study the failure of GCH under a suitable large cardinal hypothesis. More precisely, we present a theorem by Silver asserting that, if some very large cardinal exists, then GCH can fail at a measurable cardinal.

2 Iterated Forcing

The main idea behind iterated forcing is the following: given a ground model V_0, we may produce, by set forcing, a generic extension V_1; then, in V_1, we may again force and obtain a generic extension V_2 of V_1 and so on ... It turns out that the result of this process, which can be carried out into the tranfinite, can be obtained at once by employing a single notion of forcing in the ground model.
In the following, by *notion of forcing*, we mean any partial ordering with a maximum.

Definition 2.1 (Two-step iteration)**.** Let P be a notion of forcing, $\dot{Q}$ a P-name and let $\Vdash_P$ "$\dot{Q}$ is a notion of forcing".
On the set

$$\{(p, \dot{q}) \colon p \in P \text{ and } \Vdash_P \dot{q} \in \dot{Q}\}$$

define the following equivalence relation:

$$(p, \dot{q}) \sim (p', \dot{q}') \text{ iff } p = p' \text{ and } p \Vdash \dot{q} = \dot{q}'$$

Denote by $P \otimes \dot{Q}$ the quotient and define $(p, \dot{q}) \leq (p', \dot{q}')$ iff $p \leq p'$ and $p \Vdash \dot{q} \leq \dot{q}'$.

Definition 2.2. Let $\alpha \neq 0$ be an ordinal. we say that an ordered set P_α of α-sequences is an α-stage iteration iff:

- if $\alpha = 1$,

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- there is a notion of forcing Q_0 such that $p \in P_\alpha$ iff $p(0) \in Q_0$
- $p \leq q$ iff $p(0) \leq q(0)$ for every $p, q \in P_\alpha$

- if $\alpha = \beta + 1 > 1$,
  - $P_\beta := \{p|\beta \colon p \in P_\alpha\}$ is a β-stage iteration;
  - there exists a P_β-name $\dot{Q}_\beta$ such that $\Vdash_{P_\beta}$"$\dot{Q}_\beta$ is a notion of forcing" and $p \in P_\alpha$ iff $p|\beta \in P_\beta$ and $\Vdash_{P_\beta} p(\beta) \in \dot{Q}_\beta$;
  - $p \leq q$ iff $p|\beta \leq q|\beta$ and $p|\beta \Vdash p(\beta) \leq q(\beta)$ for every $p, q \in P_\alpha$;

- if α is a limit ordinal,
  - $\forall \beta < \alpha$ P_β is a β-stage iteration;
  - $(\dot{1}_\beta \colon \beta < \alpha) \in P_\alpha$, where $\dot{1}_\beta$ is the maximal element in $\dot{Q}_\beta$
  - if $\beta < \alpha$, $p \in P_\beta$, $q \in P_\alpha$ and $p \leq q|\beta$, then $\exists r \in P_\alpha$ such that $r|\beta = p$ and $r(\gamma) = q(\gamma)$ $\forall \gamma \in [\beta, \alpha)$
  - if $p, q \in P_\alpha$, $p \leq q$ iff $\forall \beta \in [1, \alpha)$ $p|\beta \leq q|\beta$

The sets Q_α are sometimes called *factors.*
Observe that,if α is a limit ordinal, P_α is not fully determined by $(P_\beta \colon \beta < \alpha)$.

Definition 2.3. For α limit ordinal,

- P_α is the *direct limit* of $(P_\beta \colon \beta < \alpha)$ iff $p \in P_\alpha$ iff $(\exists \beta < \alpha) p|\beta \in P_\beta$ and $(\forall \gamma \in [\beta, \alpha)) p(\gamma) = \dot{1}_\gamma$
- P_α is the *inverse limit* of $(P_\beta \colon \beta < \alpha)$ iff $(\forall \beta < \alpha)$ $p|\beta \in P_\beta$

Definition 2.4. For $\alpha \geq 1$, an α-stage iteration P_α is an iteration with *Easton support* iff for every limit ordinal $\gamma \leq \alpha$, P_γ is a direct limit if γ is regular and an inverse limit otherwise.

It turns out that, under suitable assumptions, an $(\alpha+\beta)$-stage iteration can be split into the product of an α-stage and a β-stage iteration.

Lemma 2.1 (The Factor Lemma)**.** *Let $P_\alpha + \beta$ be an $\alpha + \beta$-stage iteration with factors $(Q_\gamma \colon \gamma < \alpha + \beta)$ such that, at limit stages, only direct and inverse limits are taken. If, $\forall \xi \leq \beta$ limit ordinal with $cf(\xi) \leq |P_\alpha|$, $P_\alpha + \xi$ is an inverse limit, then*

$$P_{\alpha+\beta} \cong P_\alpha \otimes \dot{P}_\beta^{(\alpha)},$$

where $P_\beta^{(\alpha)}$ is the $\beta-$stage iteration obtained by letting $P_1^{(\alpha)} = \dot{Q}_\alpha$ and, for $\gamma \in [2, \beta]$, defining $P_\gamma^{(\alpha)}$ from $(P_\beta^{(\alpha)} \colon \beta < \gamma)$ as $P_{\alpha+\gamma}$ is defined from $(P_{\alpha+\beta} \colon \beta < \gamma)$.

The following a property of forcing notions will be used later.

Definition 2.5. A notion of forcing P is λ-directed closed (λ-dc) if whenever $D \subseteq P$ is directed and $|D| \leq \lambda$, there exists $p \in P$ such that $(\forall d \in D) p \leq d$.

Lemma 2.2.

- *If P is λ-dc and $\Vdash_P$ "$\dot{Q}$ is λ-dc", then $P \otimes \dot{Q}$ is λ-dc.*
- *If $cf(\alpha) > \lambda$, P_α is a direct limit and $\forall \beta < \alpha$ P_β is λ-dc, then P_α is λ-dc.*
- *If for all $\beta < \alpha$ limit ordinals P_β is either a direct or inverse limit and it is inverse when $cf(\beta) \leq \lambda$ and the factors $\dot{Q}_\beta$ are λ-dc,then P_α is λ-dc*

3 Large cardinals, forcing and elementary embeddings

Definition 3.1. Let k be an infinite cardinal:

1. k is a *Mahlo cardinal* iff it is inaccessible and the set of all regular cardinals below k is stationary;
2. k is a measurable cardinal iff there is a normal, k-complete, non-principal ultrafilter on k (i.e. a *measure* on k)
3. k is λ-supercompact iff there is a definable elementary embedding $j\colon V \to M$ such that $crit(j) = k$, $j(k) > \lambda$ and $M^\lambda \subseteq M$. k is supercompact iff it is λ-supercompact for every λ.

These notions are ordered in increasing logical strength, i.e. 3. $\to$ 2. $\to$ 1.

k-stage iterations, with k Mahlo, present an interesting property.

Proposition 3.1. *If k is a Mahlo cardinal and $(\forall \beta < k)|P_\beta| < k$ and, for $\beta < k$, P_β is a direct limit whenever β inaccessible, then P_k is k-cc.*

The following proposition offers a method to check if it is possible to lift an elementary embedding from ground models to generic extensions.

Proposition 3.2. *Let $j\colon M \to N$ be an elementary embedding between transitive models of ZFC. Let $P \in M$ be a notion of forcing, G M-P-generic and H N-j(P)-generic. The following are equivalent:*

1. $(\forall p \in G) j(p) \in H$;
2. *there exists an elementary embedding $j^+ : M[G] \to N[H]$ that extends j and such that $k^+(G) = H$*

Proof. 2.$\to$1. is evident. To show 1.$\to$2., let $j''(G) \subseteq H$ and define

$$j^+(\dot{\tau}^G) := j(\dot{\tau})^H$$

Clearly, j^+ extends, j and, using the elementarity of j, it is easy to prove that j^+ is a well-defined elementary embedding and $k^+(G) = H$. $\square$

4 Failure of GCH at a measurable cardinal

Theorem 4.1 (Silver)**.** *If there exists a supercompact cardinal k, then there is a forcing extension in which k is a measurable cardinal and $2^k > k^+$.*

Proof. Assume $2^k = k^+$.
Let $P = P_{k+1}$ a (k+1)-stage iteration with Easton support, obtained from factors $\dot{Q}_\alpha$ defined as follows:

- If α is not an inaccessible cardinal, $\dot{Q}_\alpha = \{1\}$
- If α is inaccessible, $\dot{Q}_\alpha$ names $Add(\alpha, \alpha^{++})_{V^{P_\alpha}}$

Let G be V-P-generic; since, by Lemma 2.1, there exists $\dot{Q}_k$ such that $P \cong P_k \otimes \dot{Q}_k$, there are G_k V-P_k-generic and H_k $V[G_k]$-Q_k-generic such that

$$V[G] = V[G_k][H_k]$$

. Since k is a Mahlo cardinal, by proposition 3.1, P_k is k-cc, so $k^{V[G_k]}$ is a regular cardinal. As $\dot{Q}_\alpha^{V^{P_k}} = Add(\alpha, \alpha^{++})_{V^{P_k}}$, $V[G] \Vdash$"k is regular and $2^k = k{+}{+}$".

Let $\lambda = k^{++}$ and let $j \colon V \to M$ be the elementary embedding whose existence is granted by the definition of λ-supercompactness (3.1). Clearly $|P| = \lambda$, so, since $M^\lambda \subseteq M$, $P \in M$, thus G is M-P-generic.
We state a useful, not difficult lemma.

Lemma 4.2.

$$(M[G])^\lambda \cap V[G] \subseteq M[G]$$

It can be easily seen that Lemma 2.1 can be applied to $j(P)$, obtaining

$$j(P) \cong j(P)_{k+1} \otimes j(P)^{(k+1)}_{j(k)+1}$$

Note that $j(P)_{k+1} = P_{k+1} = P$, so, if we denote by $\dot{Q}$ the second factor, we obtain

$$j(P) \cong P \otimes \dot{Q},$$

where, since $\dot{Q}$ is obtained from λ-dc factors (i.e either $\{1\}$ or names for notions of the form $Add(\alpha, \alpha^{++})$), by Lemma 2.2, $\dot{Q}$ is λ-dc and so is $Q = \dot{Q}^G$.
Let $p \in P$; we can see $j(p)$ as a couple $(s, \dot{q})$ with $s \in P$ and $\dot{q} \in \dot{Q}$. If we write p as $(p_\xi \colon \xi < k+1)$, from the definition of P we infer that $\exists \xi_0 < k$ such that $p_\xi = 1$ $\forall \xi \in [\xi_0, k)$. Being $k = crit(j)$, we then have that, if $j(p) = (p'_\xi \colon \xi < j(k)+1)$, then $p'_\xi = 1$ $\forall \xi \in [\xi_0, j(k))$; so, in particular, $p'_k = 1$, while $p'_\xi = p_\xi$ $\forall \xi < k$. It follows that $s = j(p)|(k+1) = (p|k)^\frown 1 \geq p$. Therefore if $p \in G$, then $s \in G$.

Let

$$D = \{q \in Q \colon \exists p \in G \text{ such that } j(p) = (s, \dot{q}) \text{ and } q = (\dot{q})^G\}$$

As $|P| = \lambda$, by Lemma 4.2, $j|P \in M$, thus $D \in M[G]$.
Now, from G being directed, it follows that D is directed, but $|D| \leq |P| = \lambda$ and, being Q λ-dc, there exists some $a \in Q$, called *master condition*, such that $a \leq q$ for all $q \in D$.
Let H be a V[G]-Q-generic filter containing a. If we define

$$K = \{(s, \dot{q})\colon s \in G \text{ and } (\dot{q})^G \in H\},$$

then M[K]=M[G][H].
If $p \in G$, then $j(p) = (s, \dot{q}))$, with $s \in P$ and $\Vdash \dot{q} \in \dot{Q}$. By the above argument, $s \in G$ and $q = (\dot{q})^G \geq a$, so $q \in H$. It follows that $(s, \dot{q}) \in K$ and $j''(G) \subseteq K$. By Lemma 3.2, j can be extended to an elementary embedding

$$j^+\colon V[G] \to M[K]$$

We can define the non-principal, k-complete ultrafilter

$$U = \{X \subseteq k\colon k \in j(X)\}$$

By Lemma 4.2, Q is λ-closed in $V[G]$, so forcing with H does not add any new λ-sequence. Being $|U| = \lambda$, $U \in V[G]$.

□

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