Homological dimensions

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Finitistic dimension

Definition (Finitistic dimension)

The big finitistic dimension of a ring R is defined by

$$Fd\ R = \sup\{pd\ M \mid M \in R-Mod,\ pd\ M < \infty\}$$

The little finitistic dimension of R is defined similarly

$$fd\ R = \sup\{pd\ M \mid M \in R-mod,\ pd\ M < \infty\}$$

Other finitistic dimensions can also be defined.

Finitistic dimension conjecture

Let A be a finite dimensional algebra. Then:

1. $fd\ A = Fd\ A$.
2. $fd\ A < \infty$.

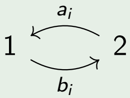
Counterexample 1

Definition

A ring R is **semiprimary** if $R/J(R)$ is semisimple, and $J(R)$ is nilpotent.

Example (by Kirkman-Kuzmanovich)

Take the quiver Γ:



And the relations ρ:

- $b_i a_j b_l = 0$ for all i, j, l,
- $a_i b_{i+l} - a_{i+l} b_{i+l} = 0$ for $l \geq 1$,
- $a_i b_j = 0$ for $i > j$,
- $b_i a_i = 0$ for all i.

Counterexample 1

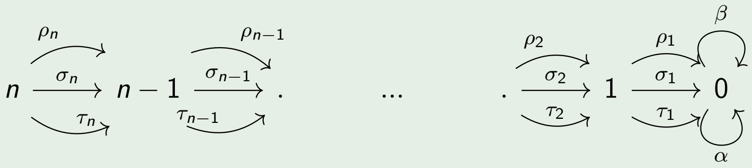
Then the following properties hold for $A = k\Gamma/\langle\rho\rangle$:

- A basis for A over k is $\{e_i, a_i, b_i, a_i b_i, b_i a_j$ for $i \neq j$, $a_i b_i a_j$ for $i \neq j\}$
- A is a semiprimary ring
- We can calculate the right annihilators of $a_i, b_i, a_i b_i$.
  - $rannih\ a_i = b_1 A \oplus \cdots \oplus b_{i-1} A \oplus e_2 A$
  - $rannih\ b_i = a_i A \oplus a_1 b_1 A \oplus \cdots \oplus a_{i-1} b_{i-1} A \oplus e_1 A$
  - $rannih\ a_i b_i = rannih\ b_i$
- Using the exact sequence $0 \to rannih\ x \to A \to xA \to 0$ we can calculate that $pd(a_1 A) = 0$, $pd(b_1 A) = 1$, $pd(a_2 A) = 2$,..., so $fd(A) = \infty$

Counterexample 2 (Zimmerman-Huisgen)

Example (SmalØ)

Let Γ_n be the quiver



let k be any field, and let Λ_n be the path algebra of Γ_n over k modulo the ideal generated by the following relations:

- α^2, β^2, $\alpha\beta$, $\beta\alpha$
- $\alpha\rho_1$, $\alpha\sigma_1$, $\beta\tau_1$
- $x_i y_{i+1}$ for $i = 1, 2, ..., n-1$ and $x \neq y; x, y \in \{\rho, \sigma, \tau\}$
- $x_i x_{i+1} - y_i y_{i+1}$ for $i = 1, 2, ..., n-1$ and $x, y \in \{\rho, \sigma, \tau\}$

Λ_n is finite dimensional algebra. $Fd\ \Lambda_n = n$ and $fd\ \Lambda_n = 1$

Finite dimensional monomial algebras

Definition

Let Γ be a finite directed graph, $k\Gamma$ the path algebra of Γ over field k, and I the ideal generated by paths of length at least two. Then $k\Gamma/\langle I\rangle$ is a **monomial algebra**.

Theorem (Igusa-Zacharia)

Let Λ be a monomial algebra over a field k. Let M be a Λ-module with finite injective dimension. Then

$$idM \leq \dim_k rad\Lambda$$

Igusa-Todorov functions

Lemma (Fitting's lemma)

Let M be a f.g. module over a Noetherian ring R and let $f : M \to M$ be an endomoprhism of M. Then for any submodule X of M there is an integer n so that f sends $f^m(X)$ isomorphically onto $f^{m+1}(X)$ for all $m \geq n$. Let $\eta_f(X)$ denote the smallest value of $n \geq 0$.

Igusa-Todorov functions

Let $K(\Lambda)$ be the quotient of the free abelian group generated by isomorphism classes $[M]$ of modules M in Λ-mod module the relations.

1. $[C] = [A] + [B]$ if $C \simeq A \oplus B$.
2. $[P] = 0$ for P projective.

Let $L[M] = [\Omega M]$ where ΩM is the first syzygy of M. Since $\Omega P = 0$ for P projective, and Ω commutes with direct sums this gives a homomorphisom $L : K(\Lambda) \to K(\Lambda)$. For f. g. Λ-module M let $\langle addM \rangle$ denote the subgroup of $K(\Lambda)$ generated by all indecomposable summand of M. Let

$$\phi(M) := \eta_L \langle addM \rangle$$

and finally let

$$\psi(M) := \phi(M) + \sup\{pdX \mid pdX < \infty, X \text{direct summand of } \Omega^{\phi(M)} M\}.$$

Igusa-Todorov functions

Properties

- If M has finite projective dimension, then $\psi(M) = \phi(M) = pdM$.
- If M is indecomposable and $pdM = \infty$ then $\phi(M) = 0$.
- $\psi(A) \leq \psi(A \oplus B)$. (Also true for ϕ)
- $\psi(kM) = \psi(M)$ for $k \geq 1$. (Also true for ϕ)
- If Z is a summand of $\Omega^n M$ where $n \leq \phi(M)$ and $pdZ < \infty$ then $pdZ + n \leq \psi(M)$

Theorem

Suppose that $0 \to A \to B \to C \to 0$ *is a short exact sequence of f.g.* Λ*-modules and* C *has finite projective dimension. Then* $pdC \leq \psi(A \oplus B) + 1$.

Thank you for you attention!