Homological Dimensions

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Chapter 1

Introduction

The main focus of this study is to gain some understanding about the Finitistic Dimension Conjecture. Let us first explain what the finitistic dimension is. (based on [6])

Definition 1.1. Let $M \in R$-Mod be an R-module. The **projective dimension** of M is n, if there is a projective resolution of M of length n, and no projective resolution of length smaller than n exists, and we denote this by $\operatorname{pd} M = n$. If there is no finite projective resolution then M has infinite projective dimension ($\operatorname{pd} M = \infty$).
The **injective dimension** is defined in a similar way, being the length of the shortest injective coresolution of M, denoted by $\operatorname{id} M = n$ if finite and $\operatorname{id} M = \infty$, if no finite resolution exists.

Definition 1.2. The **(left) global dimension of a ring** R is defined as:

$$l\,gl\,dim\,R = \sup\{\operatorname{pd} M | M \in R\text{-Mod}\} = \sup\{\operatorname{id} N | N \in R\text{-Mod}\}$$

One can also define the right global dimension, and there are examples where the left and right global dimension can differ by an arbitrary amount. On the other hand, there are also rings where $l\,gl\,dim\,R = r\,gl\,dim\,R$, in this case we denote it by $gl\,dim\,R$.

The global dimension is infinite in many cases, and thus doesn't always provide a useful overview of the ring. This motivates us to define the finitistic dimension.

Definition 1.3. The big finitistic dimension of R is defined by

$$\operatorname{Fd} R = \sup\{\operatorname{pd} M \mid M \in R\text{-Mod},\ \operatorname{pd} M < \infty\}$$

The little finitistic dimension of R is defined similarly

$$\operatorname{fd} R = \sup\{\operatorname{pd} M \mid M \in R\text{-mod},\ \operatorname{pd} M < \infty\}$$

but we only take finitely generated modules into account.

Remark 1.4. One could also define finitistic dimension with injective dimensions and/or right modules, giving a total of 8 finitistic dimensions. We will only concern ourselves with the 2 finitistic dimensions we defined.

The *finitistic dimension conjecture* first appeared in 1960 in a paper by Bass.

Conjecture 1.5. *(Finitistic dimension conjecture)*
Let A be a finite dimensional algebra. Then:

1. $\operatorname{fd} A = \operatorname{Fd} A$.

2. $fdA < \infty$.

In 1988 Green, Kirkman, and Kuzmanovich proved the second conjecture for the case of finite dimensional monomial algebras, and in 1990 Igusa and Zacharia [2] also gave an upper bound, which we'll take a look at in Chapter 3.

In 1991 Zimmermann-Huisgen refuted the first conjecture. In the next chapter we will show an example where the gap between the little and big finitistic dimension can be arbitrarily large. We will also show an example provided by Kirkman and Kuzmanovich [3], that shows the second conjecture to be false in semiprimary rings.

In 1992 Igusa and Todorov[1] also proved the second conjecture for Artin algebras with a simple condition, while also introducing a new function that is very useful in the study of projective and finitistic dimensions. We will look at this function more closely in the last chapter.

Chapter 2

Counterexamples

Let us take a look at why the finitistic dimension conjecture is not true in general for semiprimary rings given by [3]. In the following example fd is going to denote the right finitistic dimension instead (but the opposite ring is a counterexample for left finitistic dimension). This is because in path algebras writing right multiplication results in the more intuitive notation of $\alpha \cdot \beta$ meaning first going through the edges α then β.

Definition 2.1. A ring R is **semiprimary** if $R/J(R)$ is semisimple, and $J(R)$ is nilpotent (where $J(R)$ is the Jacobson-radical).

It has been shown, that when $(\operatorname{rad} A)^2 = 0$ for A semiprimary ring then $\operatorname{Fd} A$ is finite. It has also been shown that if A is an Artinian ring with $(\operatorname{rad} A)^3 = 0$ then $\operatorname{fd} A < \infty$. The counterexample has $(radA)^4 = 0$, and $\operatorname{fd} A = \infty$, it is also a factor ring of a path algebra.

Example 2.2. Take the quiver Γ:

$$\begin{tikzcd} 1 \arrow[r, bend right, "b_i"'] & 2 \arrow[l, bend right, "a_i"'] \end{tikzcd}$$

And the relations ρ:

- $b_i a_j b_l = 0$ for all i, j, l,
- $a_i b_{i+l} - a_{i+l} b_{i+l} = 0$ for $l \geq 1$,
- $a_i b_j = 0$ for $i > j$,
- $b_i a_i = 0$ for all i.

Let $A = k\Gamma/\langle \rho \rangle$. Then the following properties hold for A:

1. Let e_i be the idempotent at vertex i, then a basis for A over k is $\{e_i, a_i, b_i, a_i b_i, b_i a_j$ for $i \neq j$, $a_i b_i a_j$ for $i \neq j\}$
2. There are exactly two isomorphism classes of simple right A-modules, $\operatorname{rad} A = \langle a_i, b_i \rangle, (\operatorname{rad} A)^4 = 0, A$ is a semiprimary ring, and A is graded by powers of $\operatorname{rad} A$ since the relations are homogeneous.
3. We compute right annihilators of certain elements of A :
   $\operatorname{rannih} a_1 = e_2 A$,
   $\operatorname{rannih} b_1 = a_1 A \oplus e_1 A$ (note that $a_j b_j = a_1 b_j \in a_1 A$ for $j \geq 2$),
   $\operatorname{rannih} a_2 = b_1 A \oplus e_2 A$,
   $\operatorname{rannih} b_2 = a_2 A \oplus a_1 b_1 A \oplus e_1 A$

   $\operatorname{rannih} a_1 b_1 = \operatorname{rannih} b_1$.
   For $i \geq 3$,
   $\operatorname{rannih} a_i = b_1 A \oplus \cdots \oplus b_{i-1} A \oplus e_2 A$,
   $\operatorname{rannih} b_i = a_i A \oplus a_1 b_1 A \oplus \cdots \oplus a_{i-1} b_{i-1} A \oplus e_1 A$,
   $\operatorname{rannih} a_i b_i = \operatorname{rannih} b_i$.

4. Considering the exact sequence $0 \to \operatorname{rannih} x \to A \to xA \to 0$ it follows from 3. that $\operatorname{pd}(a_1 A) = 0$, $\operatorname{pd}(b_1 A) = 1$, $\operatorname{pd}(a_2 A) = 2$,..., so $\operatorname{fd}(A) = \infty$.

The second counterexample is for the conjecture that $\operatorname{fd} A = \operatorname{Fd} A$ for finite dimensional algebra A. In this example given by Smalø [4] we construct a finite dimensional algebra Λ_n for each natural number n, such that $\operatorname{fd} \Lambda_n = 1$, but $\operatorname{Fd} \Lambda_n = n$.

The example hinges on the fact that for finite dimensional algebras Λ_1 and Λ_2 over algebraically closed field k the global dimension of $\Lambda_1 \otimes \Lambda_2$ is the sum of the global dimension of Λ_1 and the global dimension of Λ_2. Hence we also have that $\operatorname{fd} \Lambda_1 \otimes \Lambda_2 = \operatorname{fd} \Lambda_1 + \operatorname{fd} \Lambda_2$ and $\operatorname{Fd} \Lambda_1 \otimes \Lambda_2 = \operatorname{Fd} \Lambda_1 + \operatorname{Fd} \Lambda_2$

Example 2.3. Λ_n will be given as a path algebra. Let Γ_n be the quiver

$$\begin{tikzcd}
n \arrow[r, bend left, "\rho_n"] \arrow[r, "\sigma_n"] \arrow[r, bend right, "\tau_n"'] & n-1 \arrow[r, bend left, "\rho_{n-1}"] \arrow[r, "\sigma_{n-1}"] \arrow[r, bend right, "\tau_{n-1}"'] & . & \dots & . \arrow[r, bend left, "\rho_3"] \arrow[r, "\sigma_3"] \arrow[r, bend right, "\tau_3"'] & 2 \arrow[r, bend left, "\rho_2"] \arrow[r, "\sigma_2"] \arrow[r, bend right, "\tau_2"'] & 1 \arrow[r, bend left, "\rho_1"] \arrow[r, "\sigma_1"] \arrow[r, bend right, "\tau_1"'] & 0 \arrow[loop above, "\beta"] \arrow[loop below, "\alpha"]
\end{tikzcd}$$

let k be any field, and let Λ be the path algebra of Γ over k modulo the ideal generated by the following relations:

- α^2, β^2, $\alpha\beta$, $\beta\alpha$
- $\alpha\rho_1$, $\alpha\sigma_1$, $\beta\tau_1$
- $x_i y_{i+1}$ for $i = 1, 2, ..., n-1$ and $x \neq y; x, y \in \{\rho, \sigma, \tau\}$
- $x_i x_{i+1} - y_i y_{i+1}$ for $i = 1, 2, ..., n-1$ and $x, y \in \{\rho, \sigma, \tau\}$

Theorem 2.4. *For each $n \geq 1$, Λ_n as given above has $\operatorname{fd} \Lambda_n = 1$ and $\operatorname{Fd} \Lambda_n = n$*

Chapter 3

Finite dimensional monomial algebras

Definition 3.1. Let Γ be a finite directed graph, $k\Gamma$ the path algebra of Γ over field k, and I the ideal generated by paths of length at least two. Then $k\Gamma/\langle I\rangle$ is a **monomial algebra**. From here on, Λ will denote a finite-dimensional monomial algebra.

The main result of Igusa and Zacharia [2] is the following theorem:

Theorem 3.2. *Let Λ be a monomial algebra over a field k. Let M be a Λ-module with finite injective dimension. Then*

$$\operatorname{id} M \leq \dim_k \operatorname{rad} \Lambda$$

This not only shows that for finite-dimensional monomial algebras the finitistic dimension is finite, but also gives an upped bound on its size. The main idea uses the fact that since $\operatorname{id} M$ is also the largest integer n such that $\operatorname{Ext}^n_\Lambda(S, M) \neq 0$ for simple Λ-module S. A simple module's minimal projective resolution has a nice form, as there are only a finite number of indecomposable modules which occur as summands of syzygies of simple modules. (We call these modules K_γ.) Therefore after a finite number of steps, the projective resolution of S starts repeating and this leads to a bound on the dimension of $\operatorname{Ext}^*_\Lambda(S, M)$. To construct K_γ we introduce the notion of *syzygy pairs*

Definition 3.3. Let γ be a path in the quiver of Λ of length at least 1. Suppose γ starts at v and ends at w. Let S_v denote the simple module at vertex v, and let P_v be its projective cover. Then γ gives a homomorphism of Λ-modules $\gamma^* : P_w \to P_v$ which is nonzero at w. Let $K_\gamma \subseteq P_v$ be the image of this homomorphism. Then the pair (P_v, K_γ) is a **syzygy pair** for Λ.

Lemma 3.4. *Let $\gamma : v \to w$ be a path of length ≥ 1 in the quiver of Λ. Then the kernel of the induced map $\gamma^* : P_w \to P_v$ is the direct sum of the submodules K_{γ_i} of P_w where γ_i ranges over all paths satisfying the following conditions:*

1. *$\gamma_i : w \to u_i$ is a path starting at w.*
2. *$\gamma\gamma_i : v \to u_i$ contains exactly one (minimal) zero relation ending at u_i.*

From this lemma we can define the following:

Definition 3.5. If (P_v, K_γ) is a syzygy pair, let $\Omega^1(P_v, K_\gamma)$ denote the set of all syzygy pairs (P_w, K_{γ_i}) where the γ_i are as given in 3.4. Then, for each pair (P_w, K_{γ_i}) of $\Omega^1(P_v, K_\gamma)$ we construct $\Omega^1(P_w, K_{\gamma_i})$ by applying 3.4 and we denote by $\Omega^2(P_v, K_\gamma)$ the union of all $\Omega^1(P_w, K_{\gamma_i})$. Inductively, let $\Omega^n(P_v, K_\gamma)$ be the union of the sets $\Omega^{n-1}(P_w, K_{\gamma_i})$ for $n \geq 2$.

Definition 3.6. We say that a syzygy pair (P, K) is periodic if (P, K) is isomorphic to an element of $\Omega^n(P, K)$ for some $n \geq 1$. The smallest such n is called the period of (P, K). We say that (P, K) is virtually periodic if (P, K) is isomorphic to an element of $\Omega^n(P', K')$ for some periodic pair (P', K') and some $n \geq 1$.

Then the following lemma makes all these definitions come together to help prove the main result:

Lemma 3.7. *Let* (P_0, K_0) *be any syzygy pair, and let* $n \geq \dim_k \operatorname{rad} \Lambda$*. Then every element of* $\Omega^n(P_0, K_0)$ *is virtually periodic.*

Chapter 4

Artin Algebras/Igasa-Todorov functions

Let Λ be an Artin algebra (an algebra over a commutative artin ring R that is finitely generated as and R-module). We will define a function that is useful for determining projective dimensions, but first we need Fitting's Lemma:

Lemma 4.1. *Let M be a f.g. module over a Noetherian ring R and let $f : M \to M$ be an endomoprhism of M. Then for any submodule X of M there is an integer n so that f sends $f^m(X)$ isomorphically onto $f^{m+1}(X)$ for all $m \geq n$. Let $\eta_f(X)$ denote the smallest value of $n \geq 0$. If Y is a submodule of X then $\eta_f(Y) \leq \eta_f(X)$.*
If R is an Artin algebra and $X = M$ there is a direct sum decomposition $X = Y \oplus Z$ so that $Z = \ker f^m$ and $Y = \operatorname{im} f^m$ for all $m \geq \eta_f(X)$

Let $K(\Lambda)$ be the quotient of the free abelian group generated by isomorphism classes $[M]$ of modules M in Λ-mod module the relations.

1. $[C] = [A] + [B]$ if $C \simeq A \oplus B$.
2. $[P] = 0$ for P projective.

Let $L[M] = [\Omega M]$ where ΩM is the first syzygy of M. Since $\Omega P = 0$ for P projective, and Ω commutes with direct sums this gives a homomorphisom $L : K(\Lambda) \to K(\Lambda)$. For f. g. Λ-module M let $\langle addM \rangle$ denote the subgroup of $K(\Lambda)$ generated by all indecomposable summand of M. Let

$$\phi(M) := \eta_L \langle addM \rangle$$

and finally let

$$\psi(M) := \phi(M) + \sup\{\operatorname{pd} X \mid \operatorname{pd} X < \infty, X \text{direct summand of } \Omega^{\phi(M)} M\}.$$

From the definitions and Fitting's Lemma, the following properties hold:

Statement 4.2.

1. *If M has finite projective dimension, then $\psi(M) = \phi(M) = \operatorname{pd} M$.*
2. *If M is indecomposable and $\operatorname{pd} M = \infty$ then $\phi(M) = 0$.*
3. *$\phi(A) \leq \phi(A \oplus B)$.*
4. *$\psi(A) \leq \psi(A \oplus B)$.*
5. *$\phi(kM) = \phi(M)$ for $k \geq 1$.*

6. *$\psi(kM) = \psi(M)$ for $k \geq 1$.*
7. *If Z is a summand of $\Omega^n M$ where $n \leq \phi(M)$ and $\operatorname{pd} Z < \infty$ then $\operatorname{pd} Z + n \leq \psi(M)$*

The main theorem that results in the finiteness of the (little) finitistic dimension with simple conditions is the following:

Theorem 4.3. *Suppore that $0 \to A \to B \to C \to 0$ is a short exact sequence of f.g. Λ-modules and C has finite projective dimension. Then $\operatorname{pd} C \leq \psi(A \oplus B) + 1$.*

With the right choices for A, B, and C we can easily get the following theorems:

Theorem 4.4. *Suppose that Λ is an artin algebra with $(\operatorname{rad} \Lambda)^3 = 0$ then*

$$\operatorname{fd} \Lambda \leq \psi(\Lambda/\operatorname{rad} \Lambda \oplus \Lambda/(\operatorname{rad} \Lambda)^2) + 2$$

Theorem 4.5. *If $\operatorname{repdim} \Lambda \leq 3$ then $\operatorname{fd} \Lambda < \infty$.*

Remark 4.6. $\operatorname{repdim} \Lambda \leq n$ if there is a f.g. module X such that $gl\,dim \operatorname{End}_\Lambda(X)^{op} \leq n$ and $addX$ contains all projective and injective Λ-modules. The representation dimension is always finite and many cases have repdim $= 3$, but there are examples with higher representation dimensions.

These ϕ and ψ functions turned out to be very useful in the study of finitistic dimensions among many other things lead to the notion of n-Igusa-Todorov algebras.[5]

Definition 4.7. The Artin algebra Λ is an n-Igusa-Todorov algebra if there is a module $V \in \operatorname{mod} \Lambda$ such that for any module M there exists an exact sequence

$$0 \to V_1 \to V_0 \to \Omega^n(M) \to 0$$

where $V_0, V_1 \in addV$. Such a module V is said to be an n-Igusa-Todorov module.

As you might expect the following theorem is true:

Theorem 4.8. *If A is an n-Igusa-Todorov algebra then $\operatorname{fd} A$ is finite.*

Many algebras are Igusa-Todorov algebras:

Statement 4.9. *The following following algebras A are 0-Igusa-Todorov:*

1. $\operatorname{repdim} A \leq 3$
2. *A is stably hereditary*
3. *A is a special biserial algebra*
4. *A is the extension of an iterated tilted algebra*
5. *A is a left glued algebra*
6. *A is a tilted algbra*
7. *A is a strict laura algebra*
8. *A is an algebra with radical square zero*

Chapter 5

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