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# Cooperative infinite games

Directed studies 1

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# 1 Introduction

In the mathematical camps of Lajos Pósa, the students meet many fun problems of cooperative games. As a student, I really enjoyed solving them, even though my main interest was not necessarily combinatorics. During my years at university, I visited these camps as a helper and talked with other helpers about possible questions, that can be asked connected to these camps. The main direction of the questions was to include some kind of infinity in the games. After solving many questions and seeing that this topic is indeed surprising and interesting, I held six lectures about (infinite) games in a one-week camp. In this thesis, I list the questions we asked and the results we managed to prove. Some of our work has been already asked or known, see [1], [2] and [3]. In [2], one can find consistency results connected to the axiom of choice. In this thesis, we will work in ZFC.

# 2 Games with hats

## 2.1 Hat-games with the original rules

First, let us state the original problem from the camp (which is usually given to students in grade 8).

**Problem 2.1.** There are 10 prisoners standing in a line, all facing towards the same end of the line. Each prisoner is given a hat which is either red or blue. Each prisoner can see all of those prisoners' hats who are in front of them. They guess the colour of their own hat in the order they are standing, starting with the prisoner who sees everyone else. They hear each other's guesses and can use this information to guess their own colour. The question is to determine the maximum number of correct guesses that can be guaranteed, and to give a strategy achieving it. The prisoners can fix a strategy before they get their hats, but must not communicate after that (except hearing each other's guesses).

**Proposition 2.2.** *We state that the answer for Problem 2.1 is that it can be guaranteed that at most* 1 *prisoner gives an incorrect guess. More generally, having* $k > 1$ *prisoners in the line and having* $n > 1$ *possible hat-colours, it is possible to guarantee that only the starting prisoner may make an incorrect guess.*

*Proof.* Let us convert the $n$ colours for the $n$ different residues modulo $n$, so basically we can look at the problem as having hats being the different residues. Now the first prisoner guesses the residue of the sum of all the other hats. He may not succeed, but then the next prisoner can calculate the residue of the difference of the heard guess and the sum of all hats ahead of them. This difference is indeed the hat of this prisoner. Now the next prisoners sum not only the seen hats, but also all the heard hats and subtract this sum from the first heard guess, resulting in their hat. This way, only the first person may guess incorrectly. Of course, the starting prisoner's guess cannot be guaranteed, as he does not get any information from the other hats, meaning that it is not possible to guarantee the correctness of all guesses. □

From now on, every variant will include infinity in a way. For the easier formalization, we now state the general form of the problem.

**Problem 2.3.** Suppose that prisoners stand on a subset $S \neq \emptyset$ of the real line. Each prisoner is given a hat which has a colour chosen from a set $C$ with (either finite or infinite) cardinality $\kappa \geq 2$. Note, that $C$ is also known for the prisoners before handing out the hats. Each prisoner can see those prisoners' hats, who stand on a greater number. They guess the colour of their own hat in the same order as they are standing: each prisoner hears all the guesses of the others who stand on a smaller number and can use this information to guess their own colour. The question is to determine the maximum number of correct guesses that can be guaranteed, and to give a strategy achieving it. The prisoners can fix a strategy before they get their hats, but must not communicate after that beyond hearing each other's guesses.

**Remark 2.4.** The possible answers for bounding of the number of incorrect guesses in Problem 2.3 are the following.

- For a non-negative constant $k$, it can be guaranteed that at most $k$ prisoners are incorrect, but this does not hold for any non-negative $l < k$.
- It can be guaranteed that only finitely many prisoners are incorrect, but for any non-negative $k$, it cannot be guaranteed that at most $k$ prisoners are incorrect.
- It can be guaranteed that only countably infinitely many prisoners are incorrect but it cannot be guaranteed that only finitely many prisoners are incorrect.
- It cannot be guaranteed, that only countably many prisoners are incorrect.

**Proposition 2.5.** *For any standing position $S \neq \emptyset$ and any colour set $C$ with at least two elements, it cannot be guaranteed that all prisoners are correct.*

*Proof.* Suppose indirectly that everyone gets the same element $c_1$ as a hat, so in this case, everyone should guess $c_1$. On the other hand if we change the colour of the prisoner standing on $x$ to a different colour $c_2$ (and every other colour stays $c_1$), then still every prisoner on numbers less than $x$ should guess $c_1$. But then the prisoner on $x$ heard only $c_1$ guesses and sees only $c_1$ colours, so as in the first case, he should also guess $c_1$, which would be a mistake. □

Now we give complete solutions for Problem 2.3 in the case of some particular standing subsets $S$ and colour sets $C$. In many cases, we give strategies guaranteeing that at most 1 prisoner is incorrect. In these cases, we can always use Proposition 2.5, which shows that the answer is really 1, so we will be done after proving the correctness of the strategy.

**Theorem 2.6.** *For a fixed $n$, let $S$ be the non-negative integers not greater than $n$, furthermore let $C$ be a set with an arbitrary cardinality $\kappa$. We state that it is possible to guarantee that at most only 1 prisoner is incorrect and it cannot be guaranteed that 0 prisoners are incorrect.*

*Proof.* We give a strategy guaranteeing that only the prisoner standing on 0 may guess incorrectly. If $\kappa$ is finite, then we already see the solution in Proposition 2.2. If $\kappa$ is infinite, then it is well-known that there is a bijection between the elements of $C$ and between the possible hat-colours of the $n$ people in the front as $\kappa^n = \kappa$. Agreeing on such a bijection, the first prisoner simply guesses the colour, which is the pair of the seen $n$ hats. With this, the others immediately know their colours, they do not even have to listen to the remaining guesses. □

**Remark 2.7.** In the case $C$ being the set of positive integers, we can give another elegant strategy. The prisoner on 0 can simply say the sum of all numbers he can see (without taking any residues), and the same argument works as in Proposition 2.2.

From now on, we will use a specific equivalence-relation between the possible colour-sequences. We first describe this relation, and then continue discussing special cases of the problem.

Let $\mathcal{A}$ be the set of all possibilities of putting hats on the prisoners for a given, but arbitrary $C$ and $S$. This $\mathcal{A}$ consists of (ordered) colour-sequences $A_t$ and we will denote the real colour-sequence by $A$. Each $A_t$ has ordered pairs as elements, in which the first element denotes the real number of the prisoner which they stand, and the second denotes the colour of their hat. (Note, that the different $A_t$-s only differ in the second part of their elements.) Now let us define an equivalence-relation $\sim$ in $\mathcal{A}$, between the different $A_t$-s. We say that (for two arbitrary elements of $\mathcal{A}$) $A_1$ is equivalent to $A_2$ if and only if they differ in only finitely many elements, and we denote this by $A_1 \sim A_2$. (Note that "differs" is well-defined, as $|A_1 \backslash A_2| = |A_2 \backslash A_1|$, as $|A_i| = |S|$ for all $A_i \in \mathcal{A}$.) This is really an equivalence relation, as for arbitrary $A_1, A_2, A_3 \in \mathcal{A}$, we have $|A_1 \backslash A_1| = 0$ and $|A_1 \backslash A_2| = |A_2 \backslash A_1|$ and $|A_1 \backslash A_3| \leq |A_1 \backslash A_2| + |A_2 \backslash A_3|$. It is well-known, that an equivalence-relation partitions the set into disjoint equivalence-classes. From this, we know that $\sim$ partitions $\mathcal{A}$ into disjoint equivalence-classes, let us denote the set of the classes by $\mathcal{E}$ and its elements by $E_i$. (We do not state anything about the cardinality of $\mathcal{E}$.) By the axiom of choice, let us choose an element from each equivalence-class, which will represent the whole class: for each $E_i$, choose an $R_i \in E_i$. (Of course, $R_i$ is a hat-sequence, so for every $R_i$, there exists an $A_t$, for which $R_i = A_t$.) Let us call the set of these $R_i$-s the set of representatives, and denote it by $\mathcal{R}$. By this, for every $A_t \in \mathcal{A}$, there is exactly one $R_i \in \mathcal{R}$, for which $A_t \sim R_i$. Notice, that if a prisoner knows all but finitely many hats, then they also know what is the equivalence-class of the hat-sequence, and from this they also know the representative of the sequence. We will use these notations throughout the proofs.

**Theorem 2.8.** *Let $S$ be the non-negative integers, furthermore let $C$ be $\{0, 1\}$. We state that it is possible to guarantee that at most only 1 prisoner is incorrect and it cannot be guaranteed that 0 prisoners are incorrect.*

*Proof.* We give a strategy guaranteeing that only the prisoner standing on 0 may guess incorrectly. As the prisoner on 0 sees every other prisoner's hat, he knows which sequence $R$ represents the real sequence $A$ of the other prisoners. Now he should guess 0 if $A$ and $R$ differs at even number of places and should guess 1 if they differ at odd number of places. Now we prove inductively that all the other prisoners guess correctly.

First, notice that every prisoner sees all but finitely many hats, thus everyone knows the equivalence class and the representative. So the prisoner on 1 sees every hat what 0 sees, except his own. By this, he can decide whether his hat agrees the colour which is in the representative or not, as it changes the parity of the number of differences. So the prisoner on 1 guesses correctly. From now on, suppose that everyone until $k$ guessed correctly and we want to prove that the prisoner on $k+1$ also guesses correctly. This is true, as he also knows the representative and knows that every guess before him was correct. So the same way as the prisoner on 1 could find out his colour, now the prisoner on $k+1$ also sets the parity of differences to the heard one. Thus the prisoner on $k+1$ also guessed correctly, so we completed the induction step and we are done. □

**Theorem 2.9.** *Every strategy that solves Theorem 2.8 is basically the same as the one, we gave above. Precisely, it is true, that any strategy for Theorem 2.8 is in the form of prisoner* 0 *telling the parity of the number of differences for some fixed representatives.*

*Proof.* To see this, the first observation is that the correct strategies must guarantee every guess except the first one. This is because if for a strategy there is the possibility that a prisoner is incorrect, then changing the colour on prisoner 0 (and only on him) gives the possibility of two incorrect guesses. So after the guess of the prisoner on 0, every guess should be determined and therefore no communication can be done by the others.

Now let us take an arbitrary colour-sequence, we know that everyone guessed correctly and let us denote the guess of the prisoner on 0 by $\varepsilon$. Changing the colour of an arbitrary prisoner implies that in the case of this new colour-sequence, the guess of the prisoner on 0 should be different from $\varepsilon$, basically meaning it is $1-\varepsilon$. From these new colour-sequences, we could also change only one colour, implying that the helping guess now must be $\varepsilon$. This could be repeated arbitrary times, which means that in the case of sequences which differs from the original even times, the helping guess must be $\varepsilon$, otherwise $1-\varepsilon$. This determines the strategy for the whole equivalence-class, which ends in the desired form. Of course, this reasoning works independently for all equivalence classes, so for every class, the strategy is in the desired form. The choice of representatives are independent in the classes. Note, that if we choose a sequence as a representative, and change it to another one for which the number of differences are even, the strategy will remain the same. □

**Remark 2.10.** Note, that the strategies described in Theorem 2.9 are characterized by the so-called parity functions.

**Theorem 2.11.** *Let $S$ be the non-negative integers, furthermore let $C$ be a set with an arbitrary cardinality $\kappa$. We state that it is possible to guarantee that at most only* 1 *prisoner is incorrect and it cannot be guaranteed that* 0 *prisoners are incorrect.*

*Proof.* We give a strategy guaranteeing that only the prisoner standing on 0 may guess incorrectly. There will be two cases: $\kappa$ is finite or infinite.

If $\kappa > 2$ is finite, then we can modify the given strategy in Proposition 2.2. As every prisoner knows the representative, it is enough for the prisoner on 0 to communicate the differences from the representative. The strategy is to first give a bijection between the colours and the remainders modulo $\kappa$. Then the prisoner on 0 guesses the sum of the differences from the representative modulo $\kappa$. (In other words, for every $n > 1$, he calculates the remainder of the difference of the real colour on $n$ and the representative's colour on $n$. As all but finitely many difference is 0, he can simply guess the remainder of the sum of the differences.) Now everyone else can find out the remainder of their colour's difference to the one in the representative from the earlier guesses, by the same reasoning as in Proposition 2.2.

If $\kappa$ is infinite, then there is still a representative, for which there are only finitely many differences and everyone know this representative. So it is enough for the prisoner on 0 to communicate the places of the differences and the real colours at these places. Notice, that this is a finite tuple of pairs, in which the first elements are positive integers and the second elements are the real colours. But as $\kappa$ is infinite, there is a bijection between a set with $\kappa$ elements and the finite tuples with $\kappa$ possible elements. Using this bijection, the prisoner on 0 guesses the set of bad places and their real colours, meaning that everyone will immediately know their real colour. □

**Remark 2.12.** Note that in the case of $\kappa$ being infinite, after the guess of the prisoner on 0, everyone immediately knows their colour, as there was no need to hear the other guesses. On the other hand, it is not possible to do that for the finite case, even when we only have three prisoners. (This is really an easier problem, as we could fix that every prisoner after the first three gets a specific colour.) So we state that there is no strategy, when immediately after the guess of the first prisoner, everyone knows their colour. This is true, because the third prisoner does not see anything, so for his correctness, the guess of the first prisoner is determined by only the third colour. Because of this, when changing the colour of only the second prisoner's hat, the guess should not change, which means that the second prisoner cannot find out his colour. In other words, as the guess of the first prisoner only depends on the third colour, the second prisoner does not get information about his colour.

**Theorem 2.13.** *Let $S = \{0\} \bigcup \{\frac{1}{n} : n \in \mathbb{Z}, 0 < n\}$, furthermore let $C$ be $\{0,1\}$. We state that it is possible to guarantee that at most only* 1 *prisoner is incorrect and it cannot be guaranteed that* 0 *prisoners are incorrect.*

*Proof.* We give a strategy guaranteeing that only the prisoner standing on 0 may guess incorrectly. The strategy is the same as in Theorem 2.8, so the prisoner on 0 guesses the parity of the number of differences compared to the representative.

Indirectly suppose that at least one other prisoner guessed incorrectly. If there is only one other such prisoner, then until his guess everyone guessed correctly and only finitely many prisoners are remained, so he also should have guessed correctly as he knows the correct representative.

Now suppose that there are at least two incorrect guesses. Note, that it is well-defined to look at the last two incorrect guesses, as for each prisoner, there is only finitely many after him. But then notice, that every prisoner thinks about the same representative, as all but finitely many prisoners have guessed before each prisoner. But then after the second from last incorrect guess, the parity of differences compared to the assumed representative is correct. But then the last incorrect guess would make this parity incorrect, as otherwise the second from last guess would be the opposite. This is a contradiction, as the strategy was to always have the correct parity compared to the assumed representative. □

**Theorem 2.14.** *Let $S = \{\frac{1}{n} : n \in \mathbb{Z}, 0 < n\}$, furthermore let $C$ be $\{0,1\}$. We state that it is possible to guarantee that at most only* 1 *prisoner is incorrect and it cannot be guaranteed that* 0 *prisoners are incorrect.*

*Proof.* We will use the usual equivalence-relation, giving the existence of a representative, from which only finitely many differences exist. Let $F$ be a function from the finite sets of positive numbers to the set of odd numbers, sending each set to different odd integers with the extra property that $f(H) > \max(H)$ for each finite set $H$. (Note, that $F$ can be constructed recursively by enumerating the finite sets of positive integers.) We will use this $F$, to get from a finite set of differences, to a hinting prisoner.

Every prisoner knows the representative of the heard guesses so they think about the same representative. Thus, if we assume that the representative is correct, then we could do the following strategy. Everyone should guess their colour from the representative, until the corresponding hinting prisoner on $\frac{1}{2k+1}$ guesses the opposite colour. The other prisoners hear this, so they know the set of differences compared to the representative, so every other prisoner guesses correctly.

Of course, we assumed that they guessed the correct representative. So we only need to modify this strategy, to exclude the possibility of guessing a different equivalence class. For this purpose, the prisoners on $\frac{1}{2n}$ also get a role. Their task is if until their point no hinting occurred and they see that no hinting prisoner can hint the set of differences, then they "shout", meaning that they tell the opposite colour from the heard representative. The other prisoners should notice the "shout", but should play according to the representative (as if there were no shout). Notice, that if the prisoners guess a different equivalence class, then there is a point, until all prisoners said exactly a different representative. But this would mean that infinitely many shouts occurred until that point. But this is a contradiction, as all these shouts differ from the representative, so we differ from the representative at infinitely many places, so it was not our representative. This contradiction came from the assumption of not guessing the correct equivalence class. So there is only one legit guessing sequence, the one in which everyone but at most one (the hinting) prisoner is incorrect, so we are done. □

## 2.2 Hat-games with modified rules

Now we give a few examples with hat-games, where the ground rules are a bit changed.

**Theorem 2.15.** *Let $S$ be the non-negative integers, furthermore let $C$ be $\{0, 1\}$. Also suppose that the prisoners are deaf, meaning that they cannot hear the guesses of the prisoners standing on smaller numbers. We state that it is possible to guarantee that at most only finitely many prisoners are incorrect, but for any non-negative $k$, it cannot be guaranteed that at most $k$ prisoners are incorrect.*

*Proof.* First, we give a strategy guaranteeing at most finitely many mistakes. First notice, that similarly as in Theorem 2.8, every prisoner sees all but finitely many hats, thus know the equivalence class and the representative. The strategy for everyone is to simply guess the colour which is on themselves in the representative. So basically, as everyone know the correct representative and everyone guesses by that, the guesses will form the representative itself. This means that as the colour-sequence only differs from its representative at finitely many places, the number of mistakes is also finite.

Now we prove that for any positive integer $k$, it cannot be guaranteed, that any of the first $k$ prisoners would be correct, no matter what are the colours from $k + 1$. We will prove the statement by induction. The statement is trivially true for $k = 1$, as no matter what the first prisoner sees, if he would guess correctly, only changing his colour makes the guess wrong. Now suppose that have proved the statement for $k$ and we want to prove it for $k + 1$. Now let us take an arbitrary colour-sequence. If the guess of the $k + 1$-th prisoner is correct, then change the colour of only this prisoner. This does not modify his guess, as the seen colours remain the same, meaning that now the guess is incorrect. Now as we have proven the statement for $k$, we can change the first $k$ colours such that all their guesses are wrong. But as the $k+1$-th prisoner does not see these hats, this guess remains the same, meaning that it remains wrong, completing the induction step. So we finished the proof, as we showed a possibility for every $k$, such that all of the first $k$ guesses are wrong. □

**Theorem 2.16.** *Let $S$ be the non-negative integers, furthermore let $C$ be a set with an arbitrary infinite cardinality $\kappa$. Also suppose that the prisoners are deaf, meaning that they cannot hear the guesses of the prisoners standing on smaller numbers. We state that* *it is possible to guarantee that at most only finitely many prisoners are incorrect, but for any non-negative $k$, it cannot be guaranteed that at most $k$ prisoners are incorrect.*

*Proof.* First, we give a strategy guaranteeing at most finitely many mistakes. Notice, that the exact same strategy as in Theorem 2.15, also works here. This is because everyone knows the representative here also, and can guess the colour in the representative. Thus their guesses will be the whole representative, but as there are only finitely many differences, there are only finitely many mistakes.

Now we prove that for any positive integer $k$, it cannot be guaranteed, that any of the first $k$ prisoners would be correct, no matter what are the colours from $k+1$, finishing the proof. Investigate the problem, when we also tell the prisoners that from the $\kappa$ colours, we will only use two colours as hats and we also specify these two colours. This is an easier task for them, as we did not give any restriction for their guesses, just told information about the hats used. We state that even in the case of this much easier game, they cannot guarantee for any positive integer $k$, that any of the first $k$ guesses are correct. This is because if anyone would guess a colour which is not from the specified two, then it is just an immediate mistake and as they are deaf, no information is communicated with that. (This reasoning would not work in the original problems, as more colours imply more possibilities for giving information with the guess.) This means that we can suppose that every guess is from the specified two. But now we reduced our problem to the one in Theorem 2.15, which we have already solved. □

**Theorem 2.17.** *Let $S = [0, 1]$, furthermore let $C$ be a set with an arbitrary infinite cardinality $\kappa$. Also suppose that the prisoners are deaf, meaning that they cannot hear the guesses of the prisoners standing on smaller numbers. We state that it is possible to guarantee that at most countably infinitely many prisoners are incorrect (but do not state anything about the possibility of guaranteeing something better). Furthermore at the same time, there exists a $t \in [0, 1)$ such that every prisoner in $[t, 1]$ guesses correctly.*

**Remark 2.18.** One could observe, that the usual strategy does not work here without modification, even for only two colours. As suppose, that the colour sets (for red and blue) are $R = (0, t]$ and $B = (t, 1]$ for each $t \in [0, 1]$ are representatives. These indeed fall into different classes, so it is well-defined to let them be representatives. We do not care about the other representatives. Also, assume that every prisoner gets blue as a colour. Then every prisoner (not caring about the 0-th) guessing blue is a valid guessing sequence. But note, that everyone guessing red (not caring about the 0-th) is also a valid guessing sequence. This shows that this strategy is not well-defined yet, we always have to check, that only one valid guessing sequence exists.

*Proof.* The idea of the following proof is taken from [1] and [3].

Let $\mathcal{F}$ denote the possible colour-orders, so equivalently let $\mathcal{F}$ be the set of functions from $[0, 1]$ to $\kappa$. Using axiom of choice, well-order $\mathcal{F}$ to get $\{f_\alpha : \alpha < |\mathcal{F}|\}$.

Also, modify the equivalence relation on $\mathcal{F}$ to the following: $f$ is equivalent to $g$ if and only if there exists a $t \in [0, 1)$ such that $f|_{[t,1]} = g|_{[t,1]}$. Note that this is indeed an equivalence relation, as if $f, g$ and $g, h$ are equivalent using $t_1$ and $t_2$ respectively, then $\max(t_1, t_2)$ proves the equivalence of $f, h$.

Let us hand out the colours, denote this by $f$. Now each prisoner can decide that which $g \in \mathcal{F}$ are possible only based on what they see. So each prisoner has a well-ordered set $\mathcal{G} \subseteq \mathcal{F}$, containing all the possible functions from their point of view. Now each prisoner chooses $f_\alpha \in \mathcal{G}$ which is the first in the well-ordering. Then each prisoner guesses their colour according to the chosen $f_\alpha$. We state that this is a good strategy. (Because of the well-ordering, it is well-defined.)

Note, that if a prisoner on $t$ guesses incorrectly, it means that he chose an $f_t$ which is actually not possible for all the prisoners standing behind him. This means that in the well-ordering, $f_t$ proceeds all the chosen functions $f_s$ for prisoners standing on $s < t$. This means that, as time flies, the (incorrectly chosen) functions which resulted in an incorrect guess, form a descending chain in the well-ordering. But then with the natural isomorphism, the real numbers of the incorrect guesses also form a well-ordered set. Here, the ordering is backwards, but it does not change anything. It is well-known, that the well-ordered subsets of the reals are countable. Furthermore, by this we also have the existence of $t \in [0, 1)$ such that every prisoner in $[t, 1]$ guesses correctly, as otherwise we would have an infinite decreasing chain. □

**Remark 2.19.** We can modify Theorem 2.17 such that each prisoner should guess not only their hat, but all the hats on prisoners who are standing on smaller numbers. In order to get released, the prisoner on $t$ should have a $t_0 < t$ such that his guess on the whole interval $[t_0, t]$ is correct. Simply redefining the equivalence classes gives that at most countable many prisoners are not released even in this stronger version.

**Theorem 2.20.** *For a fixed $n$, let $S$ be the non-negative integers not greater than $n$, furthermore let $C = \{0, 1\}$. Also modify the problem to include one impostor among the prisoners, who has the goal to make the most prisoners (including himself) to guess incorrectly. The others do not know who is the impostor and he knows what is going to be the strategy for the others. We state that it is possible to guarantee that at most three prisoners are incorrect.*

*Proof.* The idea is to play the exact way as in Proposition 2.2. So the strategy is that the prisoner on 0 tells the parity of the number of the red hats. If the impostor is honest, then all but the first prisoner guesses correctly. If the impostor is the first prisoner and lies, then the second prisoner guesses incorrectly. But these two guesses together switched the parity back to the correct one, so everyone else guesses correctly. Similarly, if the impostor is not the first prisoner, then his lie makes the next prisoner make a mistake. But these two mistakes again switches back the parity, so everyone else guesses correctly. So in this case, only at most two guesses are incorrect after the first prisoner.

Note, that it is not possible to guarantee at most two mistakes. This is because even for only (the first) three prisoners, it is not possible to guarantee at least one correct answer. This can be checked by investigating all the possible strategies. □

**Remark 2.21.** Note, that the previous strategy also works for $S = \mathbb{N}$ in the way that we should only play as in Theorem 2.8. This is true, as we only used that for every prisoner, there is only finitely many guessing before him.

**Conjecture 2.22.** We conjecture that in the case of $S = \mathbb{Q} \bigcap [0, 1]$ and $C = \{0, 1\}$, it is not possible to guarantee only at most one incorrect guess.

# 3 Games with boxes

I heard the following problems from Peter Simon. First, let us state the main problem.

**Problem 3.1.** There are some rooms, which contain boxes numbered by the positive integers. In each box, there is a real number (not necessarily distinct ones). The rooms are identical, meaning that for each $n \in \mathbb{N}$, the box indexed by $n$ has the same real number $r_n$ in each room. For each room, there is a prisoner, who goes inside the room, and opens the boxes in an arbitrary order. They can also choose the next box to open based on what they have seen until that. Each prisoner have to leave one box of their choice closed. Each prisoner have to guess the number in their closed box. The task is to minimize the number of incorrect guesses. The prisoners can agree on a strategy before going into the rooms, but once they go in, they must not communicate with the others.

**Theorem 3.2.** *In the case of two rooms, it is possible to guarantee that at most one guess is incorrect.*

*Proof.* We again have to work with equivalence classes. Now the set is the sequences of real numbers and the equivalence relation is again, that two sequences differ only at finitely many places. For each class, the prisoners choose a representative before going into the rooms. One should note, that the prisoners will use the classes and representatives of infinite subsequences of the boxes, not for the whole sequence.

The first prisoner opens the boxes with even indices, the second opens the boxes with odd indices. They look at the seen infinite sequence and compare it with its representative. By definition, they differ only at finitely many places. This means that there is a last index, at which the number is different from the representative. Both make the assumption that the sequence of their closed boxes is not the worse, meaning that the index of the last difference is at least at large, as the index of the other prisoner's last index of difference. For the opened boxes, let the index of the last difference be $n_i$ for $i \in \{1, 2\}$. Then the prisoners open all the remained boxes except the $n_i + 1$-th one. Now they know the representative of the other parity's boxes, so they guess for the $n_i + 1$-th box the number from the representative in this box.

This strategy is indeed good, as either $n_1 + 1 > n_2$ or $n_2 + 1 > n_1$ holds. So the prisoner which first opened the worse sequence will guess correctly. □

**Theorem 3.3.** *In the case of $n$ rooms, it is possible to guarantee that at most one guess is incorrect.*

*Proof.* We only have to slightly modify the proof of Theorem 3.2. Now each prisoner $i \in \{1, \dots, n\}$, first opens all boxes with indices not congruent to $i$ modulo $n$. For each remainder $j \neq i$, they calculate $n_j$, the index of last difference compared to the representative. Then each prisoner $i$ calculates $M_i = \max_{j \neq i}(n_j) + 1$. All of them make the assumption, that $n_i < M_i$. Then they open all but the $M_i$-th box in their congruence class. Then they guess for this box the number from the representative. As the assumption holds for at least $n - 1$ prisoners, their guesses are correct. □

**Theorem 3.4.** *In the case of countably infinitely many rooms, it is possible to guarantee that at most one guess is incorrect.*

*Proof.* We again just need to modify the proof of Theorem 3.3. Now partition the set of positive integers into infinitely many infinite subsets. The subset $N_i$ corresponds to prisoner $i$. Each prisoner opens all the boxes with indices not in $N_i$. Unfortunately the earlier used quantity $\max_{j \neq i}(n_j) + 1$ may be infinite. But it would be enough, if everyone could chose an $M_i$ based on $(N_j)_{j \neq i}$ such that $n_i < M_i + 1$ would hold for all but at most one prisoner. Note that $\max_{j \in \mathbb{N}}(n_j) + 1$ is finite, if and only if $\max_{j \neq i}(n_j) + 1$ is finite for all the prisoners. If it is finite, then we are done by choosing $M_i = \max_{j \neq i}(n_j) + 1$. In the other case, we have to modify the strategy, the following "lemma" solves this problem.

**Lemma 3.5.** *Suppose we have countably infinitely many prisoners standing in a circle. Every prisoner gets a positive integer on his head such that the set of given numbers is unbounded. The prisoners cannot see their own number, but can see all the other numbers. When the authority tells them, all should point at another prisoner at the same time. We state that it possible to to guarantee, that at most one prisoner points to a smaller number. They can agree on a strategy before getting the numbers, but cannot communicate after that.*

*Proof.* The prisoners index themselves by $\mathbb{N}$. We again use the same equivalence-relation for the sequence of positive integers. Each prisoner can see all but one numbers, so all of them knows the equivalence class and the representative. By definition, there is an $N$, which is the last difference compared to the representative. Let the representative's $i$-th element be $r_i$ and let the number on the $i$-th prisoner be $n_i$. Then for $i < N$, the $i$-th prisoner knows $N$ while the others only know that $N \geq i$. So the prisoners with $i < N$ should guess that $B_i := \max_{i \neq j \leq N}(n_j) + 1 > n_i$ while everyone else should guess that $B_i := \max_{j<i}(n_j, r_i) + 1 > n_i$. Then prisoner $i$ should point at a number greater then $B_i$, the index of this number should be $M_i$. By this, everyone with $n_i = r_i$ guesses correctly, and only at most one $n_i \neq r_i$ guesses incorrectly, the one with single greatest $n_i$. $\square$

By following the strategy in Lemma 3.5, the prisoners can choose correct indices $M_i+1$, such that at most one of them chooses a sequence with last difference earlier then their sequence's last difference. $\square$

**Remark 3.6.** One should notice, that we did not use any property of the reals. So in each version, we can replace the reals by any set with cardinality $\kappa$ for any $\kappa$.

**Remark 3.7.** One could modify the boxes-problem, such that the prisoners have to leave $k \in \mathbb{N}$ boxes closed and should guess for all of them. It can be guaranteed that all but one prisoner is correct for all of their guesses. To see that, they should simply leave the boxes with indices $M_i + 1, \ldots, M_i + k$ closed before guessing in the above strategies. They already know what is the representative, so they can guess for these boxes. As the assumption still holds for all but one prisoner, they are going to guess correctly.

**Remark 3.8.** One could come up with the idea, that this should imply that each prisoner guesses correctly with a positive probability, which is trivially not true. (Similar reasonings could have come up also at the deaf-versions of the hat-problem.) The mistake in these paradoxes is that the strategies are not measurable functions, so calculating probabilities does not make sense.

**Problem 3.9.** What happens, when we have uncountably many prisoners? For example the case of continuum prisoners with countably many boxes sounds interesting. And what if we have both the number of prisoners and boxes continuum? We have not worked on these questions yet, but it is planned in the future.

# References

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