The Hungarian Roulette Problem - Directed Studies 1

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December 20, 2024

Abstract

In this semester, I explore the Hungarian Roulette problem in which participants shoot at each other at a recurring signal until exactly only one or no participant stands. I provide a brief overview of the already known results including the distribution that models the number of players standing after a single round, the probability that in the end there remains a winning standing player, and an existing formula constructed with heuristic reasoning which models logarithmic asymptotic periodicity for this probability.

I explore and summarize the available results and graph the distribution of the number of players remaining after a single turn to gain understanding of recursions for the Hungarian Roulette and connected problems. I write up the formula for the expected recursion depth and provide a lower/upper bound on the expected depth by constraining the number of standing players at the end of the Roulette. I then present two equivalent formulas for the probability of the number of standing players taking a certain fixed value some time during the Roulette.

Kivonat

Ebben a félévben megvizsgálom a Magyar Rulett problémát, amiben a résztvevők egy ismétlődő jel hatására egymásra lőnek, ameddig pontosan csak egy résztvevő vagy egy résztvevő sem marad állva. Egy rövid áttekintést nyújtok a már ismert eredményekről, beleértve az eloszlást, ami az egy kör után állva maradó játékosok számát modellezi, a valószínűségét annak, hogy végül megmarad egy győztes álló játékos, valamint a meglévő heurisztikus érvelés segítségével összeállított formulát, ami logaritmikus aszimptotikus periodicitást modellez ezen valószínűséghez.

Az elérhető eredményeket megvizsgálom és összegzem és ábrázolom az egy kör után maradó játékosok számának az eloszlását hogy megértsem a Magyar Rulettre és a kapcsolódó problémákra vonatkozó rekurziókat. Felírom a képletet a várható rekurzió mélységre és megadok egy alsó/felső határt a várható mélységre a Roulett végén állva maradó játékosok számának korlátozásával. Ezután bemutatok két ekvivalens képletet annak a valószínűségére, hogy az álló játékosok száma egy bizonyos rögzített vesz fel bármikor a Rulett során.

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1 Introduction

The term "Hungarian Roulette" was coined by Móri et al. to describe the following random allocation problem [2]: Let there be n players, each player with a gun. At a given signal, each player points their gun to a player different from them and fires a live bullet (multiple players might shoot the same player). Players shot are then removed from the game. While there are at least 2 players, further rounds are performed. The game lasts until there are no players or there is exactly one player left – unable to shoot.

With Waring's formula [1], the authors provide the following recursion for the number of players standing after a single round:

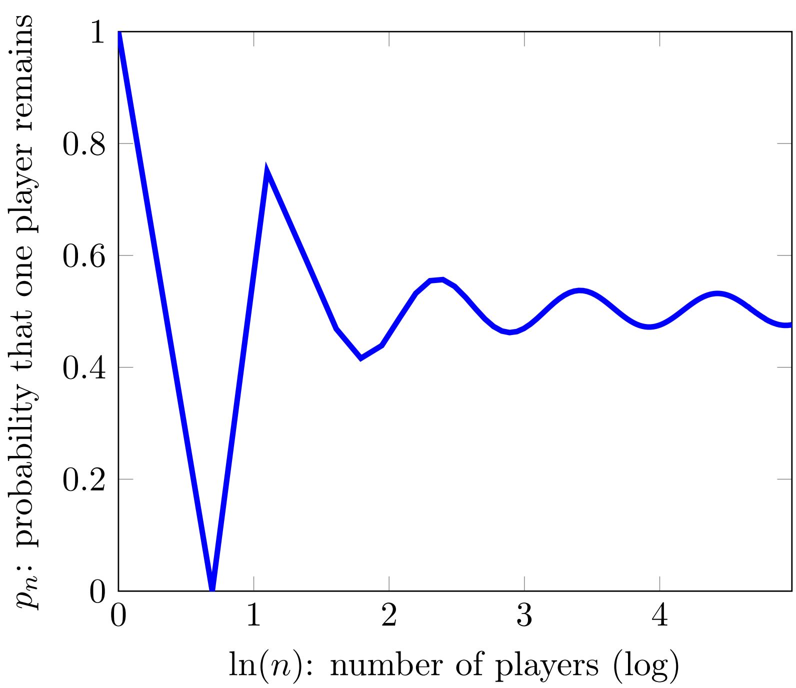
$$\mathbb{P}(\xi_n = k) = \frac{1}{(n-1)^n} \sum_{\ell=k}^{n-2} (-1)^{\ell-k} \binom{\ell}{k} \binom{n}{\ell} (n-\ell)^\ell (n-\ell-1)^{n-\ell} \tag{1}$$

The probability that the game ends with one player remaining can be calculated in the following way:

$$p_0 = 0, \quad p_1 = 1, \quad (p_2 = 0,)$$
$$p_n = \sum_{k=0}^{n-2} p_k \mathbb{P}(\xi_n = k), \quad n \geq 2 \tag{2}$$

When plotting p_n against $\ln(n)$, a sine-like periodic behavior with dampening amplitude can be observed.

Figure 1:



The authors argue that the values of p_n shown on the graph can be represented with a sine-like damped periodic function $p_n \approx c + h(\ln(n))$. It is proved that a constant κ' exists such that a lower bound can be constructed with κ' for the value of $h(M+k)$ using the value of $h(M)$ where M denotes a local maximum location:

$$h(M+k) \approx h(M) \prod_{i=0}^{k-1} \left(1 - \kappa' e^{-M-i}\right) \tag{3}$$

Note that such M may asymptotically exist. Since $\sum_{i=0}^{\infty} e^{-M-i}$ is a convergent geometric series, p_n shows asymptotic periodicity, and thus

- only converges for subsequences of n for which the fractional part of $\ln(n)$ converges (e.g., at subset of the maxima).
- does not converge in n at ∞, but instead a periodic behavior is established.

In this report, I present some intermediate results obtained from the exploration of the Hungarian Roulette problem. The rest of the report is organized as follows:

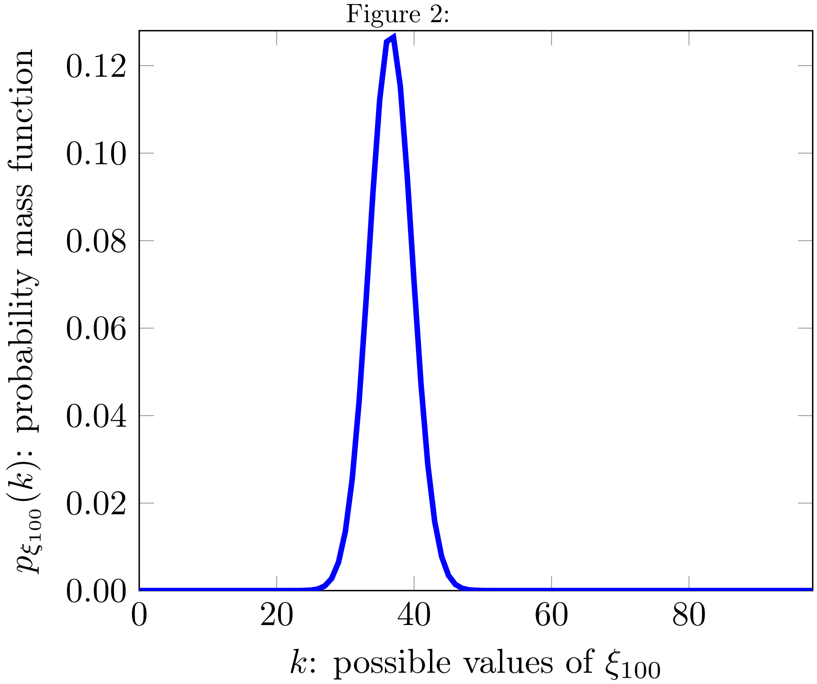
- I plot a graph of the distribution of the number of players remaining after a single turn, which provides understanding of recursions for the Hungarian Roulette and connected problems.
- I write up a formula for the expected recursion depth d_n. I show that the expected recursion depth can be upper/lower bounded by setting the constraint that the Roulette ends with a single player/no remaining players.
- I present an explicit and a recursive formula for the probability r_n^k of having k players at some point if we start the Roulette with n players. I present the graph of r_{148}^k with a brief description of what is visible on the graph.

2 Computing p_n for Large n

Since the numerical precision required for the exact computation explodes as we increase n, we cannot calculate the values at all locations for the recursive formula of p_n (see Eq. 2). Fortunately, by Taylor expansion, the authors in [2] were able to provide the approximations $\mathbb{E}\left(\frac{\xi_n}{n}\right) \approx \frac{1}{e}$ and $\mathbb{V}ar\left(\frac{\xi_n}{n}\right) \approx \frac{e-2}{e^2 n}$ for large n.

By visual inspection of the probability mass function $p_{\xi_n}(k) = P(\xi_n = k)$ for $n = 100$, we can confirm that the probability mass function takes most of its value in a narrow range of integers around $\frac{n}{e}$. Outside of this range, the value of the probability mass function is negligible.

Figure 2:



We can thus obtain a good approximation for p_n using computation methods that expand the recursion only in this short interval. Another option is Monte Carlo, which was used to plot Fig. 1.

3 Writing Up the Expected Recursion Depth d_n

The expected number of rounds d_n is given by

$$d_0 = 0, \quad d_1 = 0, \quad (d_2 = 1, \quad d_3 = 1,)$$
$$d_n = 1 + \sum_{k=0}^{n-2} d_k \; P(\xi_n = k), \quad n \geq 2 \tag{4}$$

The expected number of rounds Δ_n, such that no players remain is

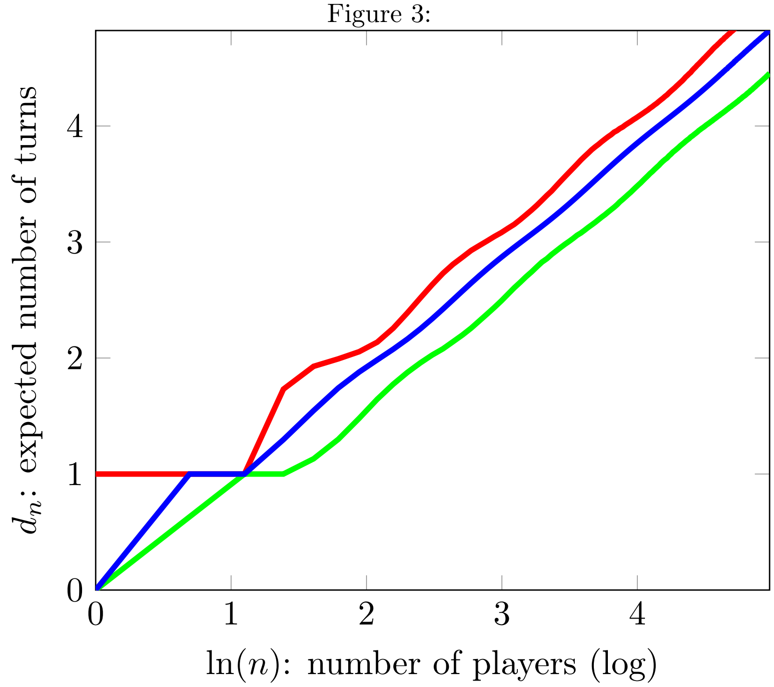
$$\Delta_0 = 0, \quad \Delta_2 = 1, \quad \Delta_3 = 1,$$
$$\Delta_n = 1 + \Delta_0 \mathbb{P}(\xi_n = 0) + \sum_{k=2}^{n-2} \Delta_k \mathbb{P}(\xi_n = k), \quad n \geq 4 \tag{5}$$

The expected number of rounds δ_n, such that one players remains is

$$\delta_1 = 0, \quad \delta_3 = 1, \quad \delta_4 = 1,$$
$$\delta_n = 1 + \delta_1 \mathbb{P}(\xi_n = 1) + \sum_{k=3}^{n-2} \delta_k \mathbb{P}(\xi_n = k), \quad n \geq 5 \tag{6}$$

We know that for all n, $\Delta_n \geq \delta_n$, since the sequence δ_n is just the sequence Δ_n shifted to the right by one. Since Δ_n and δ_n are monotonous, $\delta_n \leq \Delta_n$ for all n and since d_n is a convex combination of the two, $\delta_n \leq d_n \leq \Delta_n$. Thus, the plot of the expected depth d_n (blue) is enclosed by the upper bound Δ_n (red) and the lower bound δ_n (green).

Figure 3:



4 Probability that the player count k is reached

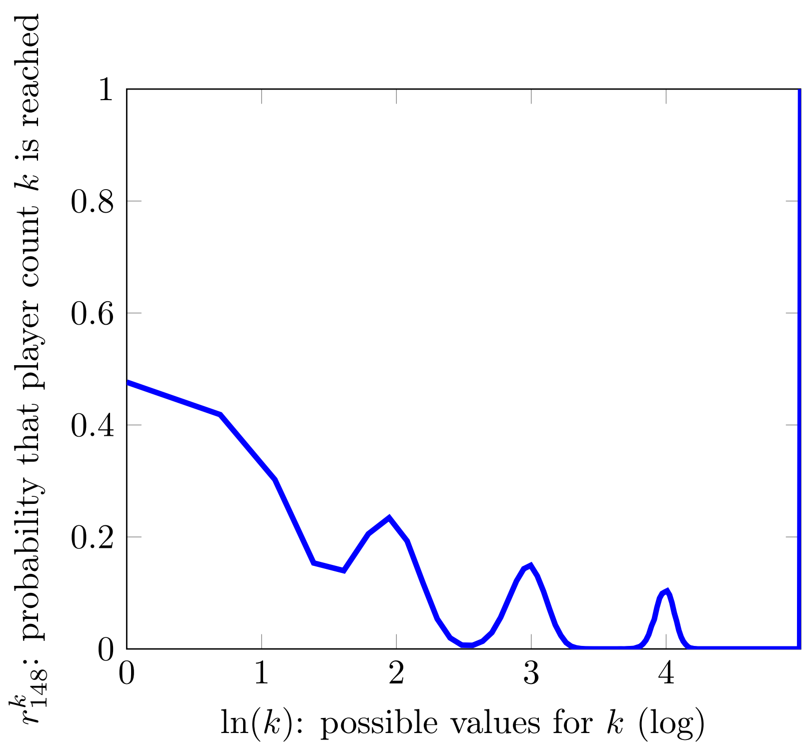
The probability r_n^k of reaching the player count k at some point can be written as a sum of disjoint probabilities:

$$r_n^n = 1$$

$$r_n^k = \sum_{k=k'_1< \ \dots \ <k'_L=n} \prod_{\ell=1}^{L-1} \mathbb{P}(\xi_{k'_{\ell+1}} = k'_\ell) = \sum_{k'=k}^{n-2} r_{k'}^k \mathbb{P}(\xi_n = k'), \quad k < n \tag{7}$$

I set the initial player count n to $148 \approx e^5$ and obtained the following graph for r_{148}^k:

Figure 4:



The four impulses shown on the graph were obtained by convolution, starting from a Kronecker delta $\delta_{148}(k)$, which is not shown on the graph. The impulses have then been juxtaposed and added.

5 Future Work

It is a future task to construct a proof of the logarithmic asymptotic periodicity of p_n. A heuristic proof is available in [2].

References

[1] William Feller. *An Introduction to Probability Theory and Its Applications, Volume 1 (revised printing, 3rd ed.)* Wiley, 1968.

[2] Tamás F. Móri and Gábor J. Székely. "Three classical probability problems: the Hungarian roulette". In: *Computatorica* 56 (2024), pp. 293–304. URL: http://ac.inf.elte.hu/Vol_056_2024/293_56.pdf.