Oracle complexivity of matroid intersection

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Introduction to matroids

Definition

Let $\mathcal{S}$ be a finite set, and let $\mathcal{F}$ be the set of the so-called *independent* subsets of $\mathcal{S}$. We call the pair $\mathcal{M} = (\mathcal{S}, \mathcal{F})$ *matroid*, if $\mathcal{F}$ satisfies the following three axoims:

- $\emptyset \in \mathcal{F}$
- If $X \subseteq Y$ and $Y \in \mathcal{F}$, then $X \in \mathcal{F}$
- For every $X \subseteq \mathcal{S}$ subset, all $K \in \mathcal{F}$, for which $K \subseteq X$ and K is maximal in X, have the same cardinality

Definition

We call r the rank function of the matroid, if
$r(X) := max(|Y| : Y \subseteq X \text{ and } Y \in \mathcal{F})$

Oracle

- There are more then *poly*($|\mathcal{S}|$) different matroids
- An oracle knows the matroid
- In a subroutine we can give the oracle any subset, and it returns yes or no, depending on if the subset is independent or not
- Rank oracle...

Matroid intersection problem

- Given two matroids $\mathcal{M}_1 = (\mathcal{S}, \mathcal{F}_1)$ and $\mathcal{M}_2 = (\mathcal{S}, \mathcal{F}_2)$, over the same ground set
- Goal: find an $X \in F_1 \cap \mathcal{F}_2$ with the highest possible cardinality
- Known results:
  - Upper bound: $\mathcal{O}(nr^{\frac{3}{2}})$
  - Lower bound: $(log_2 3)n - o(n)$ for $r = n/2$

$r \leq 1$

Theorem

Given two matroids $\mathcal{M}_1 = (\mathcal{S}, \mathcal{F}_1)$ *and* $\mathcal{M}_2 = (\mathcal{S}, \mathcal{F}_2)$ *over the same ground set, with rank at most* 1.
No algorithm that performs fewer than $2n$ *queries can solve the problem.*
However, there exists an algorithm, using $2n$ *queries.*

r$\leq$ 2

Theorem

Given two matroids $\mathcal{M}_1 = (\mathcal{S}, \mathcal{F}_1)$, $\mathcal{M}_2 = (\mathcal{S}, \mathcal{F}_2)$. *There is an algorithm to find a maximum common independence set, using* $3n - 1$ *independence oracle queries, if* $r(\mathcal{M}_1) \leq 2$ *and* $r(\mathcal{M}_2) \leq 2$.

- Find a common independent element a
- Find b, such that $\{a, b\} \in \mathcal{F}_1$ and $\{b\} \in \mathcal{F}_2$
- Find c, for which $\{a, c\} \in \mathcal{F}_2$ and $\{c\} \in \mathcal{F}_1$
- If $\{a, b\}$ and $\{a, c\}$ are not independent in both matroids, then $\{b, c\}$ is a common independent set

Rank oracle to find a basis

Theorem

Given a matroid with rank of 1*. We can find a basis, which in this case is an independent element, requiring* $\lceil log(n) \rceil$ *queries. This upper bound is also sharp.*

Theorem

Given a matroid with rank of r*. There exists an algorithm, that requires* $r \lceil log(n/r) + 1 \rceil$ *queries, at most to find a basis.*

Thank you for you attention