STANDARDLY STRATIFIED ALGEBRAS AND TILTING

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Abstract. The concept of the characteristic tilting module and of the Ringel dual for quasi-hereditary algebras is generalized for the setting of standardly stratified algebras.

In a fundamental article C.M. Ringel investigated homological properties of $\mathcal{F}(\Delta)$ (the category of modules that are filtered by standard modules) and the dually defined category $\mathcal{F}(\nabla)$ for quasi-hereditary algebras [R]. Moreover, he constructed the characteristic tilting module T which turned out to be quite important and useful in applications. He also showed that what is now called the Ringel dual $\operatorname{End}_\Lambda T$ is again quasi-hereditary. The aim of this article is to show how these results generalize to standardly stratified algebras.

More precisely, we will show that there is always a characteristic tilting module T such that the opposite of the endomorphism algebra $\operatorname{End}_\Lambda T$ is again standardly stratified. It turns out that specific properties of T characterize the case when Λ is quasi-hereditary. We will apply these results to obtain bounds for the finitistic dimensions of standardly stratified algebras, but we refer to [AHLU] for optimal bounds.

Many of our results and arguments have a clear predecessor in Ringel's article, however the subtle differences originating from the more general setting are sometimes revealing also in the quasi-hereditary situation.

We should mention that the theory of standardly stratified algebras, like that of quasi-hereditary algebras, has a natural counterpart in the theory of Lie-algebras as illustrated by the recent work of Futorny, König and Mazorchuk (see [FM1], [FM2], [FKM1], [FKM2], [FKM3]).

In the first section we have collected the necessary definitions and basic facts from the literature about standardly stratified algebras. Some statements of technical character are also included in this section. The second section then contains the main results of the paper.

After completing this paper, we have learned about the work of Platzeck and Reiten ([PR]); in particular they have also obtained parts of 2.1 and 2.2, using a different proof.

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1. Properties of standardly stratified algebras

In what follows, we recall some definitions and basic results from [D], [ADL], [DR], [AR] and [R] and derive some easy consequences.

Throughout the paper Λ will be a basic finite dimensional algebra over a field k. For simplicity we assume that k is algebraically closed, but by slightly modifying some of the dimension arguments, all our results generalize to the case of an arbitrary field. The category of finite dimensional left Λ-modules will be denoted by Λ-mod. Maps will be written on the opposite side of the scalars, thus the composition of two maps $f : X \to Y$ and $g : Y \to Z$ in Λ-mod is denoted by fg.

In the following $(\Lambda, \leq)$ will denote the algebra Λ together with a fixed ordering on a complete set $\{ e_1, \ldots, e_n \}$ of primitive orthogonal idempotents (given by the natural ordering of the indices). For $1 \leq i \leq n$ let $P(i) = \Lambda e_i$ be the indecomposable projective Λ-module and $S(i)$ the simple top of $P(i)$. The *standard module* $\Delta(i) = \Delta_\Lambda(i)$ is by definition the maximal factor module of $P(i)$ without composition factors $S(j)$ for $j > i$. We will also need particular factor modules of the standard modules: for $1 \leq i \leq n$ we denote by $\bar{\Delta}(i) = \bar{\Delta}_\Lambda(i)$ the *proper standard module*, which is the maximal factor module of $\Delta(i)$ such that the multiplicity condition

$$[\bar{\Delta}(i) : S(i)] = \dim_k \operatorname{Hom}_\Lambda(P(i), \bar{\Delta}(i)) = 1$$

holds. Note that $\operatorname{End}_\Lambda \bar{\Delta}(i) \simeq k$, so $\bar{\Delta}(i)$ is a schurian module.

We define dually the *costandard modules* $\nabla(i)$ and *proper costandard modules* $\bar{\nabla}(i)$. Thus $\nabla(i)$ is the maximal submodule of the injective envelope $I(i)$ of $S(i)$ without composition factors $S(j)$ for $i < j$, while $\bar{\nabla}(i)$ is the maximal submodule of $\nabla(i)$ that satisfies the multiplicity condition

$$[\bar{\nabla}(i) : S(i)] = \dim_k \operatorname{Hom}_\Lambda(\bar{\nabla}(i), I(i)) = 1.$$

We use the notation $\Delta = \Delta_\Lambda = \{ \Delta(1), \ldots, \Delta(n) \}$, and we define the sets $\bar{\Delta}$, ∇ and $\bar{\nabla}$ similarly.

Given a class of modules $\mathcal{C} \subseteq \Lambda$-mod, we denote by $\mathcal{F}(\mathcal{C})$ the full subcategory of Λ-mod containing all modules that are filtered by modules in $\mathcal{C}$. In particular, we shall be mostly interested in the categories $\mathcal{F}(\Delta)$, $\mathcal{F}(\bar{\Delta})$, $\mathcal{F}(\nabla)$ and $\mathcal{F}(\bar{\nabla})$.

The pair $(\Lambda, \leq)$ is called *standardly stratified* if ${}_\Lambda\Lambda \in \mathcal{F}(\Delta)$ (cf. [CPS2], [D], [ADL]; but see also [Wi] and [APT]). Note that this properly generalizes the concept of quasi-hereditary algebras where one further requires that the endomorphism algebras of the standard modules are isomorphic to k (compare for example [DR] or [CPS1] for equivalent definitions). Thus we may also note that a standardly stratified algebra $(\Lambda, \leq)$ is quasi-hereditary if and only if $\Delta(i) = \bar{\Delta}(i)$ for all $1 \leq i \leq n$.

The left-right symmetry of the conditions for quasi-hereditary algebras is replaced for standardly stratified algebras by the following theorem (see [D,2.2] or [ADL,2.2]). We denote by D the duality with respect to k.

THEOREM 1.1. *For a k-algebra $(\Lambda, \leq)$ the following are equivalent.*

(i) ${}_\Lambda\Lambda \in \mathcal{F}(\Delta)$*, i. e. Λ is standardly stratified.*

(ii) $D(\Lambda_\Lambda) \in \mathcal{F}(\bar{\nabla})$.

Next, we shall recall some basic homological facts about standard and proper costandard modules which we shall use freely later on; the statements are either well-known (cf. [DR,1.2,1.3] or [R,Ch.3]) or can be easily proved using similar methods.

LEMMA 1.2. *The following statements hold for* $(\Lambda, \leq)$.

(*i*) $\mathrm{Hom}_\Lambda\big(\Delta(i), \Delta(j)\big) = 0$ *for* $i > j$.
(*ii*) $\mathrm{Ext}^1_\Lambda\big(\Delta(i), \Delta(j)\big) = 0$ *for* $i \geq j$.
(*iii*) $\mathrm{Hom}_\Lambda\big(\bar\nabla(i), \bar\nabla(j)\big) = 0$ *for* $i < j$.
(*iv*) $\mathrm{Ext}^1_\Lambda\big(\bar\nabla(i), \bar\nabla(j)\big) = 0$ *for* $i < j$.
(*v*) $\mathrm{Hom}_\Lambda\big(\Delta(i), \bar\nabla(j)\big) = 0$ *for* $i \neq j$.
(*vi*) $\mathrm{Ext}^1_\Lambda\big(\Delta(i), \bar\nabla(j)\big) = 0$ *for every* i, j.

Moreover, if $(\Lambda, \leq)$ *is standardly stratified, then the equalities of* (*ii*), (*iv*) *and* (*vi*) *will also hold if we replace* Ext^1 *with* Ext^ℓ *for arbitrary* $\ell > 1$.

Note that the previous statements imply that for any modules $X \in \mathcal{F}(\Delta)$ and $Y \in \mathcal{F}(\bar\nabla)$ the multiplicities $[X : \Delta(i)]$ and $[Y : \bar\nabla(j)]$, giving the number of occurrences of a certain factor in a filtration, are well-defined.

We will also need the inductive building procedure of standardly stratified algebras (compare for example [D]). For this let $(\Lambda, \leq)$ be a standardly stratified algebra. Recall that $\{e_1, \dots, e_n\}$ was a complete ordered set of primitive orthogonal idempotents. For $1 \leq i \leq n$ we put $\varepsilon_i = \sum_{j=i}^n e_j$ and $\varepsilon_{n+1} = 0$. The factor algebras $\Lambda_i = \Lambda/\Lambda\varepsilon_{i+1}\Lambda$ are again standardly stratified in the induced order.

For $1 \leq i \leq n$ we define $\Delta_i = \{\Delta(1), \dots, \Delta(i)\}$ and $\bar\nabla_i = \{\bar\nabla(1), \dots, \bar\nabla(i)\}$. If we identify the Λ_i-modules with their images under the canonical inclusion functor to Λ-mod, we can summarize some basic facts concerning the relationship of the standard and costandard modules for Λ and the corresponding algebras Λ_i as follows. (The proof is obvious, hence it is omitted.)

LEMMA 1.3. *For the algebra* $(\Lambda, \leq)$ *the following statements hold.*

(*i*) $\Delta_{\Lambda_i}(j) = \Delta_\Lambda(j)$ *for* $1 \leq j \leq i \leq n$.
(*ii*) $\bar\nabla_{\Lambda_i}(j) = \bar\nabla_\Lambda(j)$ *for* $1 \leq j \leq i \leq n$.
(*iii*) $\mathcal{F}(\Delta_{\Lambda_i}) = \mathcal{F}(\Delta_\Lambda) \cap \Lambda_i\text{-mod} = \mathcal{F}(\Delta_i)$.
(*iv*) $\mathcal{F}(\bar\nabla_{\Lambda_i}) = \mathcal{F}(\bar\nabla_\Lambda) \cap \Lambda_i\text{-mod} = \mathcal{F}(\bar\nabla_i)$.
(*v*) $\mathcal{F}(\Delta_{\Lambda_i}) \cap \mathcal{F}(\bar\nabla_{\Lambda_i}) \subseteq \mathcal{F}(\Delta_\Lambda) \cap \mathcal{F}(\bar\nabla_\Lambda)$.

The next two lemmas will be useful in identifying the sets $\bar\Delta$ and ∇, respectively. Since the proofs of the two statements are similar, we shall give the proof only for the first of the lemmas.

LEMMA 1.4. *Let* $(\Lambda, \leq)$ *be given and assume that a set* $\{D(1), \dots, D(n)\}$ *of* Λ-*modules with* $\mathrm{top}\, D(i) \simeq S(i)$ *satisfies the multiplicity conditions* $[D(i) : S(i)] = 1$ *for every* i *and* $[D(i) : S(j)] = 0$ *for* $j > i$. *Assume also that* $P(i) \in \mathcal{F}\big(D(i), \dots, D(n)\big)$ *for all* $1 \leq i \leq n$. *Then* $D(i) \simeq \bar\Delta(i)$ *for all* i.

Proof. Since $\mathrm{top}\, P(i) \simeq S(i)$, and $\mathrm{top}\, D(j) \simeq S(j)$, the filtration condition $P(i) \in \mathcal{F}\big(D(i), \dots, D(n)\big)$ implies that there is an exact sequence

$$0 \to K_i \to P(i) \to D(i) \to 0,$$

with $K_i \in \mathcal{F}\big(D(i), \dots, D(n)\big)$. The multiplicity conditions on $D(i)$ and the definition of $\bar\Delta(i)$ implies that the map $P(i) \to D(i)$ factors through $\bar\Delta(i)$, hence the above sequence can be completed to the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \to & K_i & \to & P(i) & \to & D(i) & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \bar K_i & \to & \bar\Delta(i) & \to & D(i) & \to & 0 \end{array}$$

with all vertical maps surjective. If $\bar{K}_i \neq 0$ then there is an index $j < i$ such that $\operatorname{Hom}(\bar{K}_i, S(j)) \neq 0$. Consequently $\operatorname{Hom}(K_i, S(j)) \neq 0$, which contradicts the fact that $K_i \in \mathcal{F}\big(D(i), \dots, D(n)\big)$ and $\operatorname{Hom}_\Lambda\big(D(\ell), S(j)\big) = 0$ for $j < i \leq \ell$. Thus $\bar{K}_i = 0$, so $D(i) \simeq \bar{\Delta}(i)$ for every $1 \leq i \leq n$. □

LEMMA 1.5. *Let $(\Lambda, \leq)$ be given and assume that a set $\{N(1), \dots, N(n)\}$ of Λ-modules with* $\operatorname{Soc} N(i) \simeq S(i)$ *satisfies the multiplicity conditions $[N(i) : S(j)] = 0$ for $j > i$. Assume also that $I(i) \in \mathcal{F}\big(N(i), \dots, N(n)\big)$ for all $1 \leq i \leq n$, so that $N(i)$ occurs only once as a factor of a filtration of $I(i)$. Then $N(i) \simeq \nabla(i)$ for all i.*

Let us turn now to some global homological properties of the categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\bar{\nabla})$.

Recall from [AR] that a full subcategory $\mathcal{C}$ of Λ-mod which is closed under isomorphisms and direct summands is called *resolving*, if $\mathcal{C}$ is closed under extensions, contains the kernels of epimorphisms in $\mathcal{C}$ and ${}_\Lambda\Lambda \in \mathcal{C}$. The notion of *coresolving* is defined dually. The subcategory $\mathcal{C}$ is called *contravariantly finite* in Λ-mod, if every $X \in \Lambda$-mod has a *right $\mathcal{C}$-approximation*, i.e. there is a morphism $F_X \to X$ with $F_X \in \mathcal{C}$ such that the induced morphism $\operatorname{Hom}_\Lambda(C, F_X) \to \operatorname{Hom}_\Lambda(C, X)$ is surjective for all $C \in \mathcal{C}$. If X admits a right $\mathcal{C}$-approximation then it admits also a *minimal right $\mathcal{C}$-approximation*: this is a right $\mathcal{C}$-approximation which in addition is also right minimal in the sense that its restriction to any nonzero summand is nonzero. Note that for a contravariantly finite subcategory which is resolving every right approximation is surjective (cf. [AR,3.3]): indeed, the projective cover has to factor over the right approximation. The notion of a *covariantly finite subcategory* is defined dually. A subcategory of Λ-mod which is both contravariantly finite and covariantly finite is called *functorially finite*.

In the next statement we have collected some basic properties of the categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\bar{\nabla})$ for standardly stratified algebras. The statements are either known or follow easily from results in the literature.

THEOREM 1.6. *Let $(\Lambda, \leq)$ be a standardly stratified algebra. Then the following statements hold:*

(*i*) *$\mathcal{F}(\Delta)$ is a functorially finite and resolving subcategory of* Λ-mod.
(*ii*) *$\mathcal{F}(\bar{\nabla})$ is a covariantly finite and coresolving subcategory of* Λ-mod.
(*iii*) $\mathcal{F}(\Delta) = \{ X \in \Lambda\text{-mod} \mid \operatorname{Ext}^1_\Lambda\big(X, \mathcal{F}(\bar{\nabla})\big) = 0 \}$.
(*iv*) $\mathcal{F}(\bar{\nabla}) = \{ Y \in \Lambda\text{-mod} \mid \operatorname{Ext}^1_\Lambda\big(\mathcal{F}(\Delta), Y\big) = 0 \}$.

Proof. The set of standard modules Δ always satisfies $\operatorname{Ext}^1_\Lambda\big(\Delta(i), \Delta(j)\big) = 0$ for $j \leq i$, hence $\mathcal{F}(\Delta)$ is functorially finite, as stated in [R,Theorem 2]. The fact that it is also resolving will follow from [DR,1.5]. This gives (*i*). The statement of (*iv*) is contained in [ADL,3.1], using the standard k-duality. The statement of (*ii*) follows from (*i*), (*iv*) and [AR,3.3]. Finally, (*iii*) follows from (*iv*) and [AR,1.10]. □

Note that although $\mathcal{F}(\bar{\nabla})$ is not closed under kernels of epimorphisms (hence not resolving), we do have the following special case which we shall need later.

LEMMA 1.7. *Let $X \in \mathcal{F}(\bar{\nabla}_i)$ and assume that there exists a non-zero homomorphism $f \in \operatorname{Hom}_\Lambda\big(X, \bar{\nabla}(i)\big)$. Then f is surjective, and* $\operatorname{Ker} f \in \mathcal{F}(\bar{\nabla}_i)$.

Proof. Let $K = \operatorname{Ker} f$ and let $0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_s = X$ be a filtration of X with factors in $\bar{\nabla}_i$. By 1.2.(iv), we may assume that the factors isomorphic to $\bar{\nabla}(i)$ appear at the end of the filtration. If $r = \max\{\, j \mid X_j \subseteq K \,\}$, then the restriction of f to X_{r+1} will give us a non-zero map $\bar{f} : X_{r+1}/X_r \to \bar{\nabla}(i)$. Since $\operatorname{Hom}_\Lambda\big(\bar{\nabla}(j), \bar{\nabla}(i)\big) = 0$ for $j < i$, we get that $X_{r+1}/X_r \simeq \bar{\nabla}(i)$, and since $\bar{\nabla}(i)$ is schurian, we get that f is surjective.

We shall proceed by induction on the number $[X : \bar{\nabla}(i)] = t$ to show that $K \in \mathcal{F}(\bar{\nabla}_i)$. Clearly, $t > 0$, and if $t = 1$, then $K = X_{s-1}$, hence $K \in \mathcal{F}(\bar{\nabla}_i)$. So assume that $t > 1$. We may also assume that $X_{s-1} \neq K$. This gives a non-zero (hence surjective) map $\iota f : X_{s-1} \to \bar{\nabla}(i)$. Thus, we get the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & K' & \to & K & \to & \bar{\nabla}(i) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & X_{s-1} & \stackrel{\iota}{\to} & X & \to & \bar{\nabla}(i) & \to & 0. \\
 & & {\scriptstyle \iota f}\downarrow & & \downarrow{\scriptstyle f} & & & & \\
 & & \bar{\nabla}(i) & = & \bar{\nabla}(i) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Observe that $[X_{s-1} : \bar{\nabla}(i)] < [X : \bar{\nabla}(i)]$, hence by induction we get that $K' \in \mathcal{F}(\bar{\nabla}_i)$. Since K is the extension of K' by $\bar{\nabla}(i)$, we are done. □

Finally, we recall some results about the homological dimensions of the standard and proper costandard modules. The *(projectively defined) finitistic dimension* of a class of modules $\mathcal{C} \subseteq \Lambda$-mod is defined by

$$fin.dim\, \mathcal{C} = \sup\{\, proj.dim\, X \mid X \in \mathcal{C},\ proj.dim\, X < \infty \,\}.$$

The *injectively defined finitistic dimension* is defined similarly, and will be denoted by $inj.fin.dim\, \mathcal{C}$. The *finitistic dimension of the algebra* Λ is the the finitistic dimension of the whole module category Λ-mod. With this notation, we state two results from [DR,2.2] and [AHLU,3.4].

Proposition 1.8. *Let $(\Lambda, \leq)$ be a standardly stratified algebra. Then we have the following:*

(i) $\sup\{\, proj.dim\, X \mid X \in \mathcal{F}(\Delta) \,\} \leq n - 1$.
(ii) $inj.fin.dim\, \mathcal{F}(\bar{\nabla}) \leq n - 1$.

2. Main results

We keep the notation from the previous section. First we recall some additional terminology from tilting theory. A Λ-module T is called a *tilting module* if:

(a) $proj.dim_\Lambda\, T < \infty$;
(b) $\operatorname{Ext}^i_\Lambda(T, T) = 0$ for all $i > 0$;
(c) there exists an exact sequence $0 \to {}_\Lambda\Lambda \to T^0 \to T^1 \to \cdots \to T^m \to 0$ with $T^j \in \operatorname{add} T$ for all j. (Recall that $\operatorname{add} T$ is the full subcategory of Λ-mod whose objects are direct sums of direct summands of T).

The notion of a *cotilting module* is defined dually. For more details on tilting theory we refer to [H] or [AR].

Let $X \in \Lambda$-mod. We may associate to X the following useful subcategories. We denote by $X^{\perp}$ the full subcategory of Λ-mod with objects Y satisfying $\mathrm{Ext}^i_\Lambda(X,Y) = 0$ for all $i > 0$ and by ${}^{\perp}X$ the full subcategory of Λ-mod with objects Y satisfying $\mathrm{Ext}^i_\Lambda(Y,X) = 0$ for all $i > 0$. Note that ${}^{\perp}X$ is a typical example of a resolving subcategory for $X \in \Lambda$-mod. We denote by $\mathrm{fac}\, X$ the full subcategory of Λ-mod consisting of those modules Y which are epimorphic images of modules in $\mathrm{add}\, X$. Note that for a tilting module T we have $T^{\perp} \subseteq \mathrm{fac}\, T$.

Finally, if $\mathcal{C}$ is a class of modules in Λ-mod, we denote by $\widehat{\mathcal{C}}$ the full subcategory of those Λ-modules for which there exists a finite $\mathcal{C}$-resolution, i. e. an exact sequence

$$0 \to X_s \to \cdots \to X_1 \to X_0 \to X \to 0$$

with $X_i \in \mathcal{C}$ for all $0 \leq i \leq s$. Dually, one may define $\check{\mathcal{C}}$ as the full subcategory of modules with a finite $\mathcal{C}$-coresolution.

We now turn to the generalizations of Ringel's results on the characteristic tilting module for standardly stratified algebras [R].

For $X \in \Lambda$-mod we denote by $\Omega(X)$ the kernel of a projective cover of X and by $\Omega^{-}(X)$ the cokernel of an injective envelope of X. For each natural number $i \geq 1$ we set $\Omega^i(X) = \Omega\big(\Omega^{i-1}(X)\big)$ (where $\Omega^0(X) = X$) and $\Omega^{-i}(X) = \Omega^{-}\big(\Omega^{-i+1}(X)\big)$.

THEOREM 2.1. *Let $(\Lambda, \leq)$ be a standardly stratified algebra. Then there exists a tilting module ${}_\Lambda T$ such that $\mathcal{F}(\bar{\nabla}) = T^{\perp}$.*

Proof. By 1.6.(ii) we know that $\mathcal{F}(\bar{\nabla})$ is a coresolving and covariantly finite subcategory of Λ-mod. By dualizing [AR,5.5], it suffices to show that for each $X \in \Lambda$-mod there exists an exact sequence

$$0 \to X \to F_1 \to \cdots \to F_t \to 0$$

with $F_i \in \mathcal{F}(\bar{\nabla})$ for all $1 \leq i \leq n$ (i. e. that $\check{\mathcal{F}(\bar{\nabla})} = \Lambda$-mod).

Since $\mathcal{F}(\bar{\nabla})$ is coresolving, thus contains all indecomposable injective Λ-modules, it is enough to show that for each $X \in \Lambda$-mod there exists an integer d such that $\Omega^{-d}(X) \in \mathcal{F}(\bar{\nabla})$, i. e. by 1.6.$(iv)$ we should show that $\mathrm{Ext}^1_\Lambda\big(\mathcal{F}(\Delta), \Omega^{-d}(X)\big) = 0$ for some d.

If $Z \in \mathcal{F}(\Delta)$, we know by 1.8.$(i)$ that $\textit{proj.dim}_\Lambda\, Z \leq n-1$, hence $\mathrm{Ext}^n_\Lambda(Z,X) = 0$ for all $X \in \Lambda$-mod. But $\mathrm{Ext}^n_\Lambda(Z,X) \simeq \mathrm{Ext}^1_\Lambda\big(Z, \Omega^{-n+1}(X)\big)$, hence $\Omega^{-n+1}(X) \in \mathcal{F}(\bar{\nabla})$, as required. This finishes the proof. □

We will call the multiplicity-free tilting module T with $T^{\perp} = \mathcal{F}(\bar{\nabla})$ the *characteristic tilting module* for a standardly stratified algebra $(\Lambda, \leq)$. Observe, that by applying k-duality, 1.1 gives the existence of the *characteristic cotilting modules* (with dual properties) for algebras Γ for which the opposite algebra is standardly stratified.

In the sequel we will show further properties of this tilting module.

PROPOSITION 2.2. *Let $(\Lambda, \leq)$ be a standardly stratified algebra and let T be the characteristic tilting module. Then we have the following:*

(i) $\mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\nabla}) = \mathrm{add}\, T$.
(ii) $\mathcal{F}(\Delta) \subseteq {}^{\perp}T$.
(iii) $\mathcal{F}(\Delta) = \check{\mathrm{add}}\, T$.
(iv) $\textit{proj.dim}_\Lambda\, T = \max\{\, \textit{proj.dim}_\Lambda\, X \mid X \in \mathcal{F}(\Delta)\,\} \leq n-1$.

Proof. Clearly, $T \in \mathcal{F}(\bar{\nabla})$, since T is a tilting module and $\mathcal{F}(\bar{\nabla}) = T^{\perp}$, by 2.1. Furthermore, by 1.6.(iii) we infer that $T \in \mathcal{F}(\Delta)$. Hence, $\operatorname{add} T \subseteq \mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\nabla})$.

Conversely, assume that $X \in \mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\nabla})$. Then, by an earlier remark, $X \in \mathcal{F}(\bar{\nabla}) = T^{\perp} \subseteq \operatorname{fac} T$. Consider a minimal $\operatorname{add} T$-approximation of X:

$$(*) \qquad 0 \to K \to \tilde{T} \to X \to 0$$

with $\tilde{T} \in \operatorname{add} T$. By Wakamatsu's Lemma (as stated in [AR,1.3]; see also [Wa]) we infer that $\operatorname{Ext}^1_\Lambda(T, K) = 0$ and so, together with the fact that $X, \tilde{T} \in T^{\perp}$, we get that $K \in T^{\perp} = \mathcal{F}(\bar{\nabla})$. Since $X \in \mathcal{F}(\Delta)$, we know by 1.6.$(iii)$ that $\operatorname{Ext}^1_\Lambda(X, K) = 0$, hence $(*)$ splits and so $X \in \operatorname{add} T$. This shows the first assertion.

The second assertion follows immediately from (i) and 1.2.

To prove (iii), let $X \in \mathcal{F}(\Delta)$. Since $\mathcal{F}(\bar{\nabla}) = T^{\perp}$ is covariantly finite and coresolving by 1.6.(ii), by Wakamatsu's Lemma we have an exact sequence $0 \to X \to F_X \to Q_X \to 0$ with $F_X \in \mathcal{F}(\bar{\nabla})$ and $\operatorname{Ext}^1_\Lambda(Q_X, \mathcal{F}(\bar{\nabla})) = 0$. Thus $Q_X \in \mathcal{F}(\Delta)$, and therefore $F_X \in \mathcal{F}(\Delta)$, since $\mathcal{F}(\Delta)$ is closed under extensions. Hence $F_X \in \mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\nabla}) = \operatorname{add} T$ by (i). Iterating this process yields an exact sequence

$$0 \to X = X_0 \longrightarrow T^0 \xrightarrow{f_0} T^1 \xrightarrow{f_1} \cdots \xrightarrow{f_{r-1}} T^r \xrightarrow{f_r} T^{r+1} \cdots$$

for each natural number r with $T^i \in \operatorname{add} T$ and $X_i = \operatorname{Ker} f_i \in \mathcal{F}(\Delta)$ for all i.

Let $d = \mathit{proj.dim}_\Lambda T$ (observe that $d < \infty$ by 1.8.(i)) and choose such a sequence with $r = d$. Then for $i > 0$ we have that $\operatorname{Ext}^i_\Lambda(T, X_d) = \operatorname{Ext}^{d+i}(T, X_0) = 0$, so by 2.1 we get $X_d \in T^{\perp} = \mathcal{F}(\bar{\nabla})$. Since by construction also $X_d \in \mathcal{F}(\Delta)$ we infer again by (i) that $X_d \in \operatorname{add} T$

The converse inclusion is clear since $\operatorname{add} T \subseteq \mathcal{F}(\Delta)$ and $\mathcal{F}(\Delta)$ is closed under kernels of epimorphisms (cf. 1.6.(i)).

To prove (iv), recall first that, as we have already observed, 1.8.(i) implies that the projective dimension of modules in $\mathcal{F}(\Delta)$ is bounded by $n-1$. So, let $t = \max\{\, \mathit{proj.dim}\, X \mid X \in \mathcal{F}(\Delta) \,\}$ and choose $X \in \mathcal{F}(\Delta)$ with $\mathit{proj.dim}\, X = t$. Consequently, $\operatorname{Ext}^t(X, \Lambda) \neq 0$. Since $T \in \mathcal{F}(\Delta)$, it is enough to show that $t \leq \mathit{proj.dim}_\Lambda T$.

So, let $d = \mathit{proj.dim}_\Lambda T$. Then it is well-known (compare for example [H,Ch.3,2.2]) that there exists an exact sequence

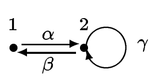
$$0 \to {}_\Lambda\Lambda \to T^0 \to T^1 \to \cdots \to T^d \to 0$$

with $T^j \in \operatorname{add} T$ for all j.

By (ii) we have $X \in {}^{\perp}T$, hence for every $s > d$ we have $\operatorname{Ext}^s(X, {}_\Lambda\Lambda) \simeq \operatorname{Ext}^{s-d}_\Lambda(X, T^d) = 0$, hence $\mathit{proj.dim}_\Lambda X = t \leq d$, as required. □

As an illustration of the previous results, let us consider the following example.

Example 2.3. Let Λ be the factor algebra kQ/I of the path algebra kQ, where

$Q :$  and $I = \langle \gamma^2, \alpha\gamma, \beta\alpha \rangle$.

The algebra has the following left and right regular representations:

$$ {}_\Lambda\Lambda = \begin{smallmatrix} & 1 & \\ & 2 & \\ 1 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 2 \end{smallmatrix}; \qquad \Lambda_\Lambda = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 \\ 1 & 2 \\ & 1 \end{smallmatrix}. $$

Here, the left standard and proper costandard modules can be described as follows:

$$ \Delta(1) = {\scriptstyle 1}, \quad \Delta(2) = \begin{smallmatrix} & 2 & \\ 1 & & 2 \end{smallmatrix}; \qquad \bar{\nabla}(1) = {\scriptstyle 1}, \quad \bar{\nabla}(2) = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}. $$

Clearly Λ is standardly stratified but not quasi-hereditary. The characteristic tilting module for Λ with the natural ordering has the following decomposition:

$$ {}_\Lambda T = {\scriptstyle 1} \oplus \begin{smallmatrix} & & 1 & & \\ & & 2 & & 1 \\ 1 & & & 2 & \end{smallmatrix}. $$

One can easily check that indeed $T \in \mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\nabla})$.

THEOREM 2.4. *Let $(\Lambda, \leq)$ be a standardly stratified algebra. Then the following statements are equivalent*

(*i*) *$(\Lambda, \leq)$ is quasi-hereditary.*
(*ii*) *$gl.dim\, \Lambda < \infty$.*
(*iii*) *The characteristic tilting module T is also cotilting and $\mathcal{F}(\Delta) = {}^\perp T$.*
(*iv*) *$\mathcal{F}(\Delta) = {}^\perp T$ for some cotilting module T.*

Proof. The equivalence of (*i*) and (*ii*) is well-known (see, for example, [PS,4.3], [Wi,1.7] and [D,2.6]).

The fact that if $(\Lambda, \leq)$ is quasi-hereditary then (*iii*) holds is contained in [R,Theorem 5, Corollary 4]. We shall, however, give a different proof for the implication (*ii*) $\Rightarrow$ (*iii*).

Thus suppose that (*ii*) holds. It is well-known that over an algebra of finite global dimension any tilting module is also a cotilting module (see, for example [H]), hence the characteristic tilting module T is also cotilting.

Since 2.2.(*ii*) gives the inclusion $\mathcal{F}(\Delta) \subseteq {}^\perp T$, we have to show only that ${}^\perp T \subseteq \mathcal{F}(\Delta)$. Let $N = \oplus_{i=1}^n \bar{\nabla}(i)$ and $d = inj.dim_\Lambda N$. Thus $inj.dim_\Lambda Y \leq d$ for all $Y \in \mathcal{F}(\bar{\nabla})$. Let us take $X \in {}^\perp T$ and consider a minimal $\mathcal{F}(\Delta)$-approximation of X, which exists and is surjective since $\mathcal{F}(\Delta)$ is functorially finite and resolving by 1.6.(*i*). So we have an exact sequence

$$ (*) \qquad 0 \to K_X \to F_X \to X \to 0 $$

with $F_X \in \mathcal{F}(\Delta)$. Now by Wakamatsu's lemma, $\mathrm{Ext}^1_\Lambda\big(\mathcal{F}(\Delta), K_X\big) = 0$, hence 1.6.(*iv*) gives that $K_X \in \mathcal{F}(\bar{\nabla}) = T^\perp$. Note that we also have $K_X \in {}^\perp T$, since $X, F_X \in {}^\perp T$ and ${}^\perp T$ is resolving.

Since $N \in \mathcal{F}(\bar{\nabla}) = T^\perp \subseteq \mathrm{fac}\, T$, by taking successive minimal $\mathrm{add}\, T$-approximations, the repeated use of Wakamatsu's lemma implies that the kernel terms will all belong to $T^\perp \subseteq \mathrm{fac}\, T$, hence for every $r > 0$ we get an exact sequence

$$ 0 \to K_r \to T^r \to \cdots \to T^2 \to T^1 \to N \to 0 $$

with $T^i \in \mathrm{add}\, T$ for all i and $K_r \in T^\perp = \mathcal{F}(\bar{\nabla})$. Let $r = d$. Thus $K_X \in {}^\perp T$ implies that we have $\mathrm{Ext}^1_\Lambda(K_X, N) \simeq \mathrm{Ext}^{r+1}(K_X, K_r)$, and this is 0, since $inj.dim_\Lambda K_r \leq d$.

So $K_X \in \mathcal{F}(\Delta)$ by 1.6.(iii), and therefore $K_X \in \mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\nabla}) = \operatorname{add} T$ by 2.2.(i). But this implies that $(*)$ splits, hence X is a direct summand of $F_X \in \mathcal{F}(\Delta)$, implying that $X \in \mathcal{F}(\Delta)$. Hence $\mathcal{F}(\Delta) = {}^{\perp}T$.

Since the implication $(iii) \Rightarrow (iv)$ is trivial, we have to show only that (iv) implies (ii). If $\mathcal{F}(\Delta) = {}^{\perp}T$ for a cotilting module T, we infer by [AR,5.5] that $\widehat{\mathcal{F}(\Delta)} = \Lambda$-mod. Recall, that this means that for every Λ-module X there exists an exact sequence

$$0 \to X_s \to \cdots \to X_1 \to X_0 \to X \to 0$$

with $X_i \in \mathcal{F}(\Delta)$ for all $0 \leq i \leq s$. Since by 1.8.(i), $proj.dim_{\Lambda} X_i < \infty$ for all i, we infer that $proj.dim_{\Lambda} X < \infty$ for every $X \in \Lambda$-mod. Hence $gl.dim\, \Lambda < \infty$. This finishes the proof of the theorem. □

We point out that the inclusion $\mathcal{F}(\Delta) \subseteq {}^{\perp}T$ will in general be a proper inclusion for a standardly stratified algebra even if the characteristic tilting module T is a cotilting module. For this let $\Lambda \neq k$ be a local selfinjective k-algebra. Then Λ is trivially standardly stratified. The unique standard module is $\Delta = {}_{\Lambda}\Lambda$ with $\bar{\nabla} = S$ the unique simple Λ-module. Then $T = {}_{\Lambda}\Lambda$, hence Λ-mod $= {}^{\perp}T$, whereas $\mathcal{F}(\Delta) = \operatorname{add} T$.

We now turn to a more detailed investigation of the characteristic tilting module T of a standardly stratified algebra $(\Lambda, \leq)$ and its endomorphism algebra $\Gamma = \operatorname{End}_{\Lambda} T$. The inductive building procedure, recalled in Section 1, induces an ordering on the indecomposable summands of T so that if we denote these summands by $T(1), \ldots, T(n)$ to reflect this ordering, then $T(i) \in \mathcal{F}(\Delta_i) \cap \mathcal{F}(\bar{\nabla}_i)$ and the characteristic tilting module T_i for the factor algebra Λ_i can be identified with $T_i = \oplus_{j=1}^{i} T(j)$. This follows immediately from 1.3 and 2.2.(i). From now on we shall fix this notation for the summands of T.

In the next lemma we have collected a few properties of the modules $T(i)$ and the images of the modules in $\mathcal{F}(\bar{\nabla})$ and $\mathcal{F}(\Delta)$ under the functors $\operatorname{Hom}_{\Lambda}(T, -)$ and $D\operatorname{Hom}_{\Lambda}(-, T)$, respectively.

Lemma 2.5. *Let $(\Lambda, \leq)$ be a standardly stratified algebra and let $T = \oplus_{i=1}^{n} T(i)$ be the characteristic tilting module. Then the following statements hold:*

(i) $[T(i) : \bar{\nabla}(j)] = 0$ *for* $i < j$ *and* $[T(i) : \bar{\nabla}(i)] > 0$ *for every* i.
(ii) $[T(i) : \Delta(j)] = 0$ *for* $i < j$ *and* $[T(i) : \Delta(i)] > 0$ *for every* i.
(iii) *For every* $1 \leq i \leq n$ *there is an exact sequence*

$$(*) \qquad 0 \to X_i \to T(i) \to \bar{\nabla}(i) \to 0$$

with $X_i \in \mathcal{F}(\bar{\nabla}_i)$.
(iv) *For every* $1 \leq i \leq n$ *there is an exact sequence*

$$(**) \qquad 0 \to \Delta(i) \to T(i) \to Y_i \to 0$$

with $Y_i \in \mathcal{F}(\Delta_{i-1})$.
(v) $\operatorname{Hom}_{\Lambda}\big(T(i), \bar{\nabla}(j)\big) = 0$ *for* $i < j$ *and* $\dim_k \operatorname{Hom}_{\Lambda}\big(T(i), \bar{\nabla}(i)\big) = 1$ *for all* i.
(vi) $\operatorname{Hom}_{\Lambda}\big(\Delta(j), T(i)\big) = 0$ *for* $i < j$.

Proof. The first two statements follow from the fact that $T(i) \in \Lambda_i$-mod but $T(i) \notin \Lambda_{i-1}$-mod.

By 1.2, we can rearrange any $\bar{\nabla}$-filtration or Δ-filtration of $T(i)$ to get the desired exact sequences $(*)$ and $(**)$, respectively, with $Y_i \in \mathcal{F}(\Delta_i)$. We still have to prove that $[Y_i : \Delta(i)] = 0$. Thus, let us assume that $[Y_i : \Delta(i)] > 0$. Then 1.2.(ii) implies that we can get an exact sequence

$$0 \to \Delta(i) \oplus \Delta(i) \to T(i) \to Y_i' \to 0$$

with $Y_i' \in \mathcal{F}(\Delta_i)$. If we apply now the functor $\mathrm{Hom}_\Lambda(-,T)$ to this sequence, by 2.2.(ii) we get the following exact sequence:

$$0 \to \mathrm{Hom}_\Lambda(Y_i', T) \to \mathrm{Hom}_\Lambda\big(T(i), T\big) \to \mathrm{Hom}_\Lambda\big(\Delta(i) \oplus \Delta(i), T\big) \to 0.$$

Observe that this is a contradiction, since $\mathrm{Hom}_\Lambda\big(T(i), T\big)$ is an indecomposable projective right $\Gamma = \mathrm{End}_\Lambda(T,T)$-module. Thus, we get that $[T(i) : \Delta(i)] = 1$, i.e. $Y_i \in \mathcal{F}(\Delta_{i-1})$.

The first statement of (v) (respectively, the statement of (vi)) follows from the fact that $T(i) \in \Lambda_i$-mod and $\bar{\nabla}(j)$ has simple socle (respectively, $\Delta(j)$ has simple top) isomorphic to $S(j)$.

To finish the proof, let us assume that $f, g \in \mathrm{Hom}_\Lambda\big(T(i), \bar{\nabla}(i)\big)$ are linearly independent over $k = \mathrm{End}_\Lambda\big(\bar{\nabla}(i)\big)$. By 1.7 both maps are surjective, with $X = \mathrm{Ker}\, f$ and $\mathrm{Ker}\, g$ in $\mathcal{F}(\bar{\nabla}_i)$, and the independence gives that $X \not\subset \mathrm{Ker}\, g$. Hence the restriction of g to X gives a non-zero (hence, by 1.7 surjective) map $\gamma : X \to \bar{\nabla}(i)$ with $\mathrm{Ker}\, \gamma \in \mathcal{F}(\bar{\nabla}_i)$. This yields the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & T(i) & \stackrel{f}{\longrightarrow} & \bar{\nabla}(i) & \longrightarrow & 0 \\
 & & \gamma\downarrow & & (g,f)\downarrow & & \| & & \\
0 & \longrightarrow & \bar{\nabla}(i) & \stackrel{\iota_1}{\longrightarrow} & \bar{\nabla}(i) \oplus \bar{\nabla}(i) & \stackrel{\pi_2}{\longrightarrow} & \bar{\nabla}(i) & \longrightarrow & 0
\end{array}$$

with ι_1 and π_2 the first inclusion and second projection, respectively. This will imply that we get a surjective map $T(i) \stackrel{(g,f)}{\longrightarrow} \bar{\nabla}(i) \oplus \bar{\nabla}(i)$, with the kernel in $\mathcal{F}(\bar{\nabla})$. Applying $\mathrm{Hom}_\Lambda(T,-)$ will, as before, yield that the epimorphic image of the indecomposable projective Γ-module $\mathrm{Hom}_\Lambda\big(T, T(i)\big)$ decomposes into the sum of two isomorphic direct summands, a contradiction.

This finishes the proof. □

We should note here that (v) does not imply that $[T(i) : \bar{\nabla}(i)] = 1$. As a matter of fact, in contrast to the quasi-hereditary situation, for standardly stratified algebras it may happen that we have $[T(i) : \bar{\nabla}(i)] > 1$.

Before formulating our main result about the *Ringel dual* $\Gamma = \mathrm{End}_\Lambda T$ for a standardly stratified algebra $(\Lambda, \leq)$, let us introduce some notation. We shall denote by F the functor $\mathrm{Hom}_\Lambda(T,-)$ and by G the functor $D\,\mathrm{Hom}_\Lambda(-,T)$, both mapping Λ-mod to Γ-mod. The left projective Γ-module $F\big(T(n-i+1)\big)$ will be denoted by $R(i)$. From now on $(\Gamma, \leq)$ will denote the algebra Γ equipped with the natural order coming from the indices of the projective modules $R(i)$. Note that this is the opposite of the order inherited from Λ and T. We shall also need the Γ-modules $F\big(\bar{\nabla}_\Lambda(n-i+1)\big) = D(i)$ and $G\big(\Delta_\Lambda(n-i+1)\big) = N(i)$.

THEOREM 2.6. *Let $(\Lambda, \leq)$ be a standardly stratified algebra and T be the characteristic tilting module with $\Gamma = \operatorname{End}_\Lambda T$. Using the notation introduced above, we have the following statements for $(\Gamma, \leq)$:*

(i) $D(i) \simeq \bar{\Delta}_\Gamma(i)$ *for* $1 \leq i \leq n$.

(ii) $N(i) \simeq \nabla_\Gamma(i)$ *for* $1 \leq i \leq n$.

(iii) The functor F induces an equivalence between $\mathcal{F}(\bar{\nabla}_\Lambda)$ and $\mathcal{F}(\bar{\Delta}_\Gamma)$.

(iv) The functor G induces an equivalence between $\mathcal{F}(\Delta_\Lambda)$ and $\mathcal{F}(\nabla_\Gamma)$.

(v) ${}_\Gamma\Gamma \in \mathcal{F}(\bar{\Delta}_\Gamma)$. *In particular, the opposite algebra Γ^{op} is standardly stratified.*

(vi) The module $T' = F\big(D(\Lambda_\Lambda)\big) \in \Gamma$-mod is a cotilting module with $\mathcal{F}(\bar{\Delta}_\Gamma) = {}^\perp T'$ and $\mathcal{F}(\bar{\Delta}_\Gamma) \cap \mathcal{F}(\nabla_\Gamma) = \operatorname{add} T'$, i. e. T' is the characteristic cotilting module of Γ.

(vii) $\Lambda \simeq \operatorname{End}_\Gamma(T')$ *and the ordering given by T' gives back the original ordering on Λ.*

Proof. To prove that $D(i) \simeq \bar{\Delta}_\Gamma(i)$, we have to show that the conditions of 1.4 are satisfied by the modules $D(i)$. Let us apply the functor F to the exact sequence $(*)$ in 2.5.*(iii)*. Then since $\mathcal{F}(\bar{\nabla}_\Lambda) = T^\perp$ by 2.1, we get an epimorphism $F\big(T(j)\big) \to F\big(\bar{\nabla}(j)\big) \to 0$, hence for $j = n - i + 1$ we get that $D(i)$ is a local module, with $\operatorname{top} D(i) \simeq \operatorname{top} R(i)$. The multiplicity conditions on $D(i)$ are implied by 2.5.*(v)*, while the filtration condition $R(i) \in \mathcal{F}\big(D(i), \dots, D(n)\big)$ will follow from the filtration properties of $T(n - i + 1)$, described in 2.5.*(i)* or *(iii)*. Observe that here we have repeatedly used the fact that the functor F is exact on short exact sequences from $\mathcal{F}(\bar{\nabla}_\Lambda)$.

A similar argument, using 1.5 and 2.5.*(ii)*, *(iv)* and *(vi)* will imply that $N(i) \simeq \nabla_\Gamma(i)$, as stated in *(ii)*.

As we have already observed, F is exact on $\mathcal{F}(\bar{\nabla}_\Lambda)$, thus by *(i)* it carries $\mathcal{F}(\bar{\nabla}_\Lambda)$ into $\mathcal{F}(\bar{\Delta}_\Gamma)$. By tilting theory (see, for example [H] or [M]) we get that F is an equivalence between the two subcategories. This proves *(iii)*.

Similarly, by 2.2.*(ii)*, the functor G is exact on $\mathcal{F}(\Delta_\Lambda)$, thus G maps $\mathcal{F}(\Delta_\Lambda)$ into $\mathcal{F}(\nabla_\Gamma)$. The fact that G is full and faithful on $\mathcal{F}(\Delta)$ follows from the fact that G is full and faithful on $\operatorname{add} T$ and the existence of a finite T-coresolution for every module in $\mathcal{F}(\Delta)$ by 2.2.*(iii)*. Finally we show that G is dense, i. e. every ${}_\Gamma Y \in \mathcal{F}(\nabla_\Gamma)$ is isomorphic to $G(X)$ for some $X \in \mathcal{F}(\Delta_\Lambda)$. First note that by tilting theory we have that ${}_\Gamma D(T)$ is a Γ-cotilting module. Hence by the dual of 2.2 we infer that $\mathcal{F}(\nabla_\Gamma) \subset D(T)^\perp$. Now consider the functor $G' = \operatorname{Hom}_\Gamma(D(T), -)$. It is easy to see that this gives an inverse to G.

To get *(v)*, one has only to observe that since $T \in \mathcal{F}(\bar{\nabla}_\Lambda)$ is mapped to $F(T) \in \mathcal{F}(\bar{\Delta}_\Gamma)$ by *(iii)*, ${}_\Gamma\Gamma \simeq F(T) \in \mathcal{F}(\bar{\Delta}_\Gamma)$; now one can use the dual of 1.1.

To prove *(vi)*, observe first that $T' \in \mathcal{F}(\bar{\Delta}_\Gamma)$ by 1.1 and *(iii)*. Next, we will show that T' is Ext-injective in $\mathcal{F}(\bar{\Delta}_\Gamma)$, i. e. we show that $\mathcal{F}(\bar{\Delta}_\Gamma) \subseteq {}^\perp T'$. For this let $Y \in \mathcal{F}(\bar{\Delta}_\Gamma)$. Then by *(iii)* there is $X \in \mathcal{F}(\bar{\nabla}_\Lambda)$ such that $Y = F(X)$. By tilting theory we infer that $\operatorname{Ext}^i_\Gamma(Y, T') \simeq \operatorname{Ext}^i_\Lambda\big(X, D(\Lambda_\Lambda)\big) = 0$ for $i > 0$. Thus T' is Ext-injective in $\mathcal{F}(\bar{\Delta}_\Gamma)$. By the dual of 1.6.*(iii)* we get that $T' \in \mathcal{F}(\nabla_\Gamma)$, hence $\operatorname{add} T' \subseteq \mathcal{F}(\bar{\Delta}_\Gamma) \cap \mathcal{F}(\nabla_\Gamma)$. On the other hand, by *(v)* and the dual of 2.1 and 2.2 we get that there exists a (basic) cotilting module T'' such that $\operatorname{add} T'' = \mathcal{F}(\bar{\Delta}_\Gamma) \cap \mathcal{F}(\nabla_\Gamma)$. Since the number of non-isomorphic indecomposable summands of T' and T'' must be the same, we get the statement of *(vi)*.

The first part of *(vii)* is straightforward since $\mathcal{F}(\bar{\nabla}_\Lambda)$ and $\mathcal{F}(\bar{\Delta}_\Gamma)$ are equivalent categories by *(iii)*, hence $\Lambda \simeq \operatorname{End}_\Lambda\big(D(\Lambda_\Lambda)\big) \simeq \operatorname{End}_\Gamma\big(F(D(\Lambda_\Lambda))\big) = \operatorname{End}_\Gamma(T')$.

Finally, observe, that $\mathrm{Hom}_\Lambda\big(T(j), I(i)\big) = 0$ if $j < i$ and $\mathrm{Hom}_\Lambda\big(T(i), I(i)\big) \neq 0$ by 2.5.(i), hence for the natural ordering of the summands of T' we have $T'(i) \simeq F\big(I(n-i+1)\big)$. This implies that after applying the tilting-cotilting procedure, we get back the original ordering on Λ.

This finishes the proof of the theorem. □

To conclude, we shall apply the previous considerations to obtain a bound for the projectively defined finitistic dimension of a standardly stratified algebra. For an optimal bound, obtained by different methods, as well as for a bound on the injectively defined finitistic dimension, we refer to [AHLU,2.1,3.1].

COROLLARY 2.7. *Let $(\Lambda, \leq)$ be a standardly stratified algebra. Then $fin.dim\,\Lambda \leq 2n-1$.*

Proof. Let $X \in \Lambda$-mod with $d = proj.dim_\Lambda X < \infty$. We may assume that $d \geq n$. Since $\mathcal{F}(\Delta)$ is contravariantly finite and resolving (cf. 1.6.(i)), Wakamatsu's Lemma, gives the existence of a short exact sequence $0 \to K_X \to F_X \to X \to 0$ with $F_X \in \mathcal{F}(\Delta)$ and $K_X \in \mathcal{F}(\bar{\nabla})$. Since $F_X \in \mathcal{F}(\Delta)$ we infer from 1.8.(i) that $proj.dim_\Lambda F_X \leq n-1$, hence $proj.dim_\Lambda K_X \leq d-1$.

Since $K_X \in \mathcal{F}(\bar{\nabla}) = T^\perp \subseteq \mathrm{fac}\,T$, and it is of finite projective dimension, repeatedly taking minimal $\mathrm{add}\,T$-approximations will yield an exact sequence

$$(*) \qquad 0 \to T^r \to \cdots \to T^1 \to T^0 \to K_X \to 0$$

with $T^i \in \mathrm{add}\,T$ for all i. Note that here we may assume that $T^r \in \mathrm{add}\,T$. Indeed, let $s = proj.dim\,K_X$ and let us denote by K_i the kernel term of the i-th approximation. Then by applying $\mathrm{Hom}_\Lambda(-, K_{s+1})$ to the short exact sequences of $(*)$, we get that $\mathrm{Ext}^1(K_s, K_{s+1}) = \mathrm{Ext}^{s+1}(K_X, K_{s+1}) = 0$. Hence K_s is in $\mathrm{add}\,T$. Let us apply now $\mathrm{Hom}_\Lambda(T, -)$ to $(*)$; this yields a finite minimal projective resolution of the Γ-module $Y_X = \mathrm{Hom}_\Lambda(T, K_X)$ where $\Gamma = \mathrm{End}_\lambda(T)$. So $proj.dim_\Gamma Y_X = r$. By 2.6.(v) we get that $Y_X \in {}_\Gamma\mathcal{F}(\bar{\Delta})$. But now the dual of 1.8.(ii) gives us that $r \leq n-1$. Since from 2.2.(iv) we have $proj.dim_\Lambda T \leq n-1$, a straightforward argument gives that $proj.dim_\Lambda K_X \leq 2n-2$, hence $d \leq 2n-1$. □

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