STRATIFYING PAIRS OF SUBCATEGORIES FOR CPS-STRATIFIED ALGEBRAS

István Ágoston[1] and Erzsébet Lukács[1]

Abstract. Two special types of module subcategories are defined over stratified algebras of Cline, Parshall and Scott. We show that for every stratified algebra there exists a (not necessarily unique) cotorsion pair of subcategories which describe to a large extent the stratification structure of the algebra. These subcategories generalize the notion of modules with standard and costandard filtration for standardly stratified and quasi-hereditary algebras.

1. Introduction

In the theory of quasi-hereditary and standardly stratified (Δ or $\bar{\Delta}$-filtered) algebras the subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\bar{\nabla})$ of modules with standard and proper costandard filtration play a crucial role (see for example [DR], [ADL1], [AHLU]). One of the key homological features of these subcategories is that they are perpendicular to each other; in fact they form a so-called cotorsion pair. Much of the structure theory and a (limited) left-right symmetry for these algebras stems from this fact. On the other hand so far no such pairing is known for the more general case of strictly stratified algebras and CPS-stratified algebras (cf. [ADL2] and [CPS]) and they also lack a reasonable structure theory.

In this article we will present a setting in which to every CPS-stratified algebra A we will associate a special cotorsion pair so that the corresponding subcategories of modules with appropriate filtration will describe the structure of projective and injective A-modules, in particular, the structure of the regular module itself.

Thus in Section 2 we will define the concept of stratifying and costratifying subcategories and describe their basic properties. We will relate these subcategories to subcategories of modules with standard or costandard filtration over standardly stratified algebras. In Section 3 we will show that by taking the perpendicular category of a stratifying subcategory we get a costratifying subcategory (and vice versa). Moreover we show that for each CPS-stratified algebra we can find a pair

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of stratifying and costratifying subcategories $\mathcal{P}$ and $\mathcal{Q}$, respectively, so that the pair $(\mathcal{P}, \mathcal{Q})$ is a hereditary cotorsion pair (this will be called a stratifying pair). For quasi-hereditary algebras this pair is given by the category of modules with standard and costandard filtration, respectively. Finally we give an example of a CPS-stratified algebra for which an infinite number of stratifying pairs exist.

2. Stratifying and costratifying subcategories

Let K be an arbitrary field and $(A, \mathbf{e})$ a basic finite dimensional K-algebra with a (linearly) ordered complete set $\mathbf{e} = (e_1, \dots, e_n)$ of primitive orthogonal idempotents and let $\varepsilon_i = e_i + \cdots + e_n$ for $1 \le i \le n$. Throughout the paper we shall be dealing with finitely generated right A-modules. In particular, $P(i) = e_i A$ will stand for the ith indecomposable projective module, $Q(i) = \operatorname{Hom}_K(Ae_i, K)$ the ith indecomposable injective module and $S(i) \simeq P(i)/\operatorname{Rad} P(i) \simeq \operatorname{Soc} Q(i)$ the corresponding simple module. The category of all finitely generated right A-modules will be denoted by mod-A. If $\mathcal{C}$ is an arbitrary class of modules in mod-A then $\mathcal{F}(\mathcal{C})$ is the full subcategory of mod-A consisting of modules M with a $\mathcal{C}$-filtration, i. e. a chain of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$ such that the factor modules M_i/M_{i-1} all belong to $\mathcal{C}$.

For $1 \le i \le n$, let us define the subclasses of modules $\mathcal{P}_i(\mathbf{e})$ as

$$\mathcal{P}_i(\mathbf{e}) = \{\, X \in \text{mod-}A \mid X \in \mathcal{F}(S(1), \dots, S(i)),\ \operatorname{Ext}^t(X, S(j)) = 0\ \forall t \ge 0,\ j < i \,\},$$

and $\mathcal{P}(\mathbf{e}) = \mathcal{F}(\mathcal{P}_1(\mathbf{e}), \dots, \mathcal{P}_n(\mathbf{e}))$. Note that the condition $\operatorname{Ext}^t(X, S(j)) = 0$ for $t \ge 0,\ j < i$ is equivalent to the fact that the projective modules in the minimal projective resolution of X all belong to $\operatorname{add}(\varepsilon_i A)$. We call the algebra $(A, \mathbf{e})$ *CPS-stratified* if $A_A \in \mathcal{P}(\mathbf{e})$, i. e. all projective modules are in $\mathcal{P}(\mathbf{e})$ (cf. [CPS], [ADL2], [ADL3]).

Dually, we define the subclasses $\mathcal{Q}_i(\mathbf{e})$ as

$$\mathcal{Q}_i(\mathbf{e}) = \{\, Y \in \text{mod-}A \mid Y \in \mathcal{F}(S(1), \dots, S(i)),\ \operatorname{Ext}^t(S(j), Y) = 0\ \forall t \ge 0,\ j < i \,\},$$

and $\mathcal{Q}(\mathbf{e}) = \mathcal{F}(\mathcal{Q}_1(\mathbf{e}), \dots, \mathcal{Q}_n(\mathbf{e}))$. Analogously to the case of $\mathcal{P}_i(\mathbf{e})$, the condition $\operatorname{Ext}^t(S(j), Y) = 0$ for $t \ge 0$, $j < i$ is equivalent to the fact that the injective modules in the minimal injective resolution of Y all belong to $\operatorname{add}(\operatorname{Hom}_K(A\varepsilon_i, K))$. Since $(A, \mathbf{e})$ is CPS-stratified if and only if $(A^{opp}, \mathbf{e})$ is CPS-stratified (as it immediately follows from [ADL3], Definition 2.1 and 2.2), all injective modules over a CPS-stratified algebra $(A, \mathbf{e})$ are in $\mathcal{Q}(\mathbf{e})$. Note that the definition implies that both $\mathcal{P}_i(\mathbf{e})$ and $\mathcal{Q}_i(\mathbf{e})$ are closed under extensions, direct summands, kernels of epimorphisms, and cokernels of monomorphisms.

For an A-module X, we denote by $T_i(X)$ the trace of the projective module $P(i) \oplus \cdots \oplus P(n)$ in X; thus $T_i(X) = X\varepsilon_i A$. In other terms, $T_i(X)$ is the unique submodule of X such that $\operatorname{Hom}(T_i(X), S(j)) = 0$ for all $j < i$, and $X/T_i(X) \in \mathcal{F}(S(1), \dots, S(i-1))$. Dually, let $R_i(X)$ be the reject of the injective module $Q(i) \oplus \cdots \oplus Q(n)$ in X, i. e. the largest submodule of X such that $R_i(X) \in \mathcal{F}(S(1), \dots, S(i-1))$. Then $R_i(X)$ is the unique submodule of X for which $R_i(X) \in \mathcal{F}(S(1), \dots, S(i-1))$ and $\operatorname{Hom}(S(j), X/R_i(X)) = 0$ for all $j < i$.

In the sequel we shall frequently make use of the following equivalence (cf. [CPS], [APT] or [ADL3]): the algebra $(A, \mathbf{e})$ is CPS-stratified if and only if for every $X, Y \in \text{mod-}(A/T_i(A))$ $(1 \le i \le n)$ and every $t \ge 0$ we have $\operatorname{Ext}^t_{A/T_i(A)}(X, Y) = \operatorname{Ext}^t_A(X, Y)$.

Lemma 2.1. *Let $X \in \mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_n)$, where $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$ for all i and take a filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_k = X$ of X with factors $Y_r = X_r/X_{r-1}$ from $\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_n$. Then X has a filtration with the same factors (up to isomorphism) but possibly in different order such that for the factors $Y'_1, \dots, Y'_k$ we have $Y'_r \in \mathcal{P}_{i_r}$ with $i_1 \geq i_2 \geq \ldots \geq i_k$.*

Proof. We use induction on k and the fact that $\mathrm{Ext}^1(\mathcal{P}_i(\mathbf{e}), \mathcal{P}_j(\mathbf{e})) = 0$ for $i > j$. □

Lemma 2.2. *Let $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$ for $1 \leq i \leq n$. Then $X \in \mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_n)$ if and only if the trace factors $T_i(X)/T_{i+1}(X)$ are in $\mathcal{F}(\mathcal{P}_i)$ for $1 \leq i \leq n$.*

Proof. If the factors $T_i(X)/T_{i+1}(X)$ are in $\mathcal{F}(\mathcal{P}_i)$ for each i then clearly $X \in \mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_n)$. For the converse, let us observe first that by Lemma 2.1, we may take a filtration $0 = X_0 \subset \ldots \subset X_k = X$ with factors $Y_r = X_r/X_{r-1} \in \mathcal{P}_{i_r}$ such that $i_1 \geq \cdots \geq i_k$. Let s be the last index such that $i_s \geq i$. Since $\mathrm{Hom}(Y_r, S(j)) = 0$ for all $r \leq s$ and $j < i$, we have $\mathrm{Hom}(X_s, S(j)) = 0$ for $j < i$, and also $X/X_s \in \mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_{i-1}) \subseteq \mathcal{F}(S(1), \dots, S(i-1))$, thus $X_s = T_i(X)$, and the statement follows □

Proposition 2.3. *Suppose that $(A, \mathbf{e})$ is CPS-stratified and let $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$ be such that $\mathcal{F}(\mathcal{P}_i)$ are closed under kernels of epimorphisms. Then $\mathcal{P} = \mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_n)$ is also closed under kernels of epimorphisms.*

Proof. We use induction on n, the number of simple modules. $\bar{A}$ will stand for the algebra $A/T_n(A)$ and in general, for $X \in$ mod-A we shall have $\bar{X} = X/T_n(X) \in$ mod-A.

Since for a CPS-stratified algebra $\mathrm{Ext}^t_{\bar{A}}(X, Y) = \mathrm{Ext}^t_{A}(X, Y)$ for all $X, Y \in$ mod-$\bar{A}$, the subclasses $\mathcal{P}_i(\mathbf{e}) \subseteq$ mod-A and $\mathcal{P}_i(\mathbf{e}') \subseteq$ mod-$\bar{A}$ will be equal for $1 \leq i \leq n-1$, where $\mathbf{e}' = (e_1, \dots, e_{n-1})$. Thus we get by induction that $\mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_{n-1})$ as a subcategory of mod-$\bar{A}$ is closed under kernels of epimorphisms, hence the same holds for $\mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_{n-1})$ as a subcategory of mod-A.

Suppose now that $0 \to X \to Y \xrightarrow{g} Z \to 0$ is exact, and $Y, Z \in \mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_n)$. It is easy to see that g_1, the restriction of g maps surjectively $T_n(Y)$ to $T_n(Z)$ and we also have an induced surjection $\bar{Y} \xrightarrow{\bar{g}} \bar{Z}$. Thus by the Snake Lemma we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Ker}\, g_1 & \to & T_n(Y) & \xrightarrow{g_1} & T_n(Z) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X & \to & Y & \xrightarrow{g} & Z & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Ker}\, \bar{g} & \to & \bar{Y} & \xrightarrow{\bar{g}} & \bar{Z} & \to & 0\,. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By Lemma 2.2, $T_n(Y), T_n(Z) \in \mathcal{F}(\mathcal{P}_n)$, and since $\mathcal{F}(\mathcal{P}_n)$ is closed under kernels of epimorphisms by assumption, $\mathrm{Ker}\, g_1 \in \mathcal{F}(\mathcal{P}_n)$. Similarly, $\bar{Y}$ and $\bar{Z}$ are in $\mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_{n-1})$, hence by induction we get that $\mathrm{Ker}\, \bar{g} \in \mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_{n-1})$. This implies that $X \in \mathcal{P} = \mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_n)$, as required. □

Let us recall that a subcategory of mod-A is called *resolving* if it is closed under extensions, direct summands and kernels of epimorphisms, and it contains all projective A-modules. Thus we have the following statement.

PROPOSITION 2.4. *If the algebra $(A, \mathbf{e})$ is CPS-stratified, then $\mathcal{P}(\mathbf{e})$ is a resolving subcategory of* mod-A.

Proof. From the definition of $\mathcal{P}(\mathbf{e})$ it is clear that it is closed under extensions. Observe that $\mathcal{P}_i(\mathbf{e}) = \mathcal{F}(\mathcal{P}_i(\mathbf{e}))$ is closed under direct summands, hence the fact that $\mathcal{P}(\mathbf{e})$ is closed under direct summands easily follows from Lemma 2.2, using that $T_i(X \oplus Y) = T_i(X) \oplus T_i(Y)$. Next, since $\mathcal{P}_i(\mathbf{e})$ is closed under kernels of epimorphisms, Lemma 2.3 implies that the same holds for $\mathcal{P}(\mathbf{e})$. Finally $A_A \in \mathcal{P}(\mathbf{e})$ holds for a CPS-stratified algebra, so all projective modules are in $\mathcal{P}(\mathbf{e})$. □

DEFINITION. Let $\mathcal{P}$ be a resolving subcategory of mod-A. We say that $\mathcal{P}$ is a *stratifying subcategory* if there are $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$ for $1 \leq i \leq n$ such that $\mathcal{P} = \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$.

LEMMA 2.5. *If $\mathcal{P} = \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ is a stratifying subcategory with $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$ then $\mathcal{F}(\mathcal{P}_i) = \mathcal{P} \cap \mathcal{P}_i(\mathbf{e})$.*

Proof. We only need to prove that $\mathcal{P} \cap \mathcal{P}_i(\mathbf{e}) \subseteq \mathcal{F}(\mathcal{P}_i)$. Suppose $X \in \mathcal{P} \cap \mathcal{P}_i(\mathbf{e})$. Then $X \in \mathcal{F}(S(1), \ldots, S(i))$ implies that $T_{i+1}(X) = 0$. Hence $X \in \mathcal{P}_i(\mathbf{e})$ gives $X = T_i(X) = T_i(X)/T_{i+1}(X)$, so $X \in \mathcal{F}(\mathcal{P}_i)$ by Lemma 2.2. □

PROPOSITION 2.6. *A subcategory $\mathcal{P} = \mathcal{F}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ with $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$ is stratifying if and only if each $\mathcal{F}(\mathcal{P}_i)$ is closed under direct summands and kernels of epimorphisms, and $T_i(A_A)/T_{i+1}(A_A) \in \mathcal{F}(\mathcal{P}_i)$.*

Proof. Suppose $\mathcal{P}$ is stratifying. Then $\mathcal{F}(\mathcal{P}_i) = \mathcal{P} \cap \mathcal{P}_i(\mathbf{e})$ by Lemma 2.5, hence it is closed under the given operations, since so are $\mathcal{P}$ and $\mathcal{P}_i(\mathbf{e})$. Furthermore, Lemma 2.2 and $A_A \in \mathcal{P}$ implies that $T_i(A)/T_{i+1}(A) \in \mathcal{F}(\mathcal{P}_i)$. In the opposite direction, the last condition implies that $(A, \mathbf{e})$ is CPS-stratified, i. e. all projective A-modules are in $\mathcal{P}$. Clearly, $\mathcal{P}$ is closed under extensions and by Proposition 2.3 $\mathcal{P}$ is closed under kernels of epimorphisms. Finally to prove that $\mathcal{P}$ is closed under direct summands we can follow a similar argument as in the proof of Proposition 2.4. □

We shall also need the duals of the previous statements. The proofs follow by straightforward dualization.

LEMMA 2.7. *Let $X \in \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$, where $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$ for all i and take a filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_k = X$ of X with factors $Y_r = X_r/X_{r-1}$ from $\mathcal{Q}_1 \cup \ldots \cup \mathcal{Q}_n$. Then X has a filtration with the same factors (up to isomorphism) but possibly in different order such that for the factors $Y'_1, \ldots, Y'_k$ we have $Y'_r \in \mathcal{Q}_{i_r}$ with $i_1 \leq i_2 \leq \ldots \leq i_k$.*

LEMMA 2.8. *Let $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$ for $1 \leq i \leq n$. Then $X \in \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ if and only if the factors $R_{i+1}(X)/R_i(X)$ are in $\mathcal{F}(\mathcal{Q}_i)$ for $1 \leq i \leq n$.*

Proposition 2.9. *Suppose that* $(A, \mathbf{e})$ *is CPS-stratified and let* $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$ *be such that* $\mathcal{F}(\mathcal{Q}_i)$ *are closed under cokernels of monomorphisms. Then* $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ *is also closed under cokernels of monomorphisms.*

A subcategory of mod-A is called *coresolving* if it is closed under extensions, direct summands and cokernels of monomorphisms, and it contains all injective A-modules. Thus we have the following statement.

Proposition 2.10. *If the algebra* $(A, \mathbf{e})$ *is CPS-stratified, then* $\mathcal{Q}(\mathbf{e})$ *is a coresolving subcategory of* mod-A.

Definition. Let $\mathcal{Q}$ be a coresolving subcategory of mod-A. We say that $\mathcal{Q}$ is a *costratifying subcategory* if there are $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$ for $1 \leq i \leq n$ such that $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$.

Lemma 2.11. *If* $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ *is a costratifying subcategory with* $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$ *then* $\mathcal{F}(\mathcal{Q}_i) = \mathcal{Q} \cap \mathcal{Q}_i(\mathbf{e})$.

Proposition 2.12. *A subcategory* $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ *with* $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$ *is costratifying if and only if each* $\mathcal{F}(\mathcal{Q}_i)$ *is closed under direct summands and cokernels of monomorphisms, and* $R_{i+1}(D({}_AA))/R_i(D({}_AA)) \in \mathcal{F}(\mathcal{Q}_i)$.

Note that $(A, \mathbf{e})$ is a CPS-stratified algebra if and only if there exists a stratifying subcategory in mod-A, or equivalently, if there exists a costratifying subcategory in mod-A. In fact, for a CPS-stratified algebra $\mathcal{P}(\mathbf{e})$ is the largest stratifying and $\mathcal{Q}(\mathbf{e})$ is the largest costratifying subcategory. Examples of minimal stratifying and costratifying subcategories will be provided by subcategories of modules with standard and costandard filtration.

Let us first recall the definition of standard and costandard modules. For a given algebra $(A, \mathbf{e})$ the *standard module* $\Delta(i)$ is defined as $\Delta(i) = P(i)/T_{i+1}(P(i))$ for $1 \leq i \leq n$. Dually, the costandard module $\nabla(i)$ is defined as $\nabla(i) = R_{i+1}(Q(i))$. The *proper standard module* $\bar{\Delta}(i)$ is the largest quotient of $\Delta(i)$ such that the composition multiplicity $[\bar{\Delta}(i) : S(i)] = 1$. Similarly, the *proper costandard module* $\bar{\nabla}(i)$ is the largest submodule of $\nabla(i)$ such that $[\bar{\nabla}(i) : S(i)] = 1$. Then with the notation $\Delta = \{\, \Delta(1), \ldots, \Delta(n) \,\}$, $\bar{\Delta} = \{\, \bar{\Delta}(1), \ldots, \bar{\Delta}(n) \,\}$, $\nabla = \{\, \nabla(1), \ldots, \nabla(n) \,\}$ and $\bar{\nabla} = \{\, \bar{\nabla}(1), \ldots, \bar{\nabla}(n) \,\}$ we get the subcategories $\mathcal{F}(\Delta)$, $\mathcal{F}(\bar{\Delta})$, $\mathcal{F}(\nabla)$ and $\mathcal{F}(\bar{\nabla})$. The algebra $(A, \mathbf{e})$ is called Δ-*filtered* if $A_A \in \mathcal{F}(\Delta)$ and $\bar{\Delta}$-*filtered* if $A_A \in \mathcal{F}(\bar{\Delta})$. The algebra $(A, \mathbf{e})$ is *standardly stratified* if either $(A, \mathbf{e})$ or $(A^{opp}, \mathbf{e})$ is Δ-filtered. It is easy to see that $(A^{opp}, \mathbf{e})$ is Δ-filtered if and only if $D({}_AA) \in \mathcal{F}(\nabla)$ and it is well-known (cf. [D]) that these conditions are equivalent to $A_A \in \mathcal{F}(\bar{\Delta})$, i. e. that $(A, \mathbf{e})$ is $\bar{\Delta}$-filtered.

Proposition 2.13. *Let* $(A, \mathbf{e})$ *be CPS-stratified, and* $\mathcal{P}$ *a stratifying subcategory. Then* $\mathcal{F}(\Delta) \subseteq \mathcal{P} \subseteq \mathcal{P}(\mathbf{e})$. *Furthermore,* $\mathcal{F}(\Delta)$ *is a stratifying subcategory if and only if* $A_A \in \mathcal{F}(\Delta)$. *Dually, for every costratifying subcategory* $\mathcal{Q}$ *we have* $\mathcal{F}(\nabla) \subseteq \mathcal{Q} \subseteq \mathcal{Q}(\mathbf{e})$, *moreover* $\mathcal{F}(\nabla)$ *is a costratifying subcategory if and only if* $D({}_AA) \in \mathcal{F}(\nabla)$ *(i. e.* A *is* $\bar{\Delta}$*-filtered).*

Proof. Since $P(i) \in \mathcal{P}$, we get $\Delta(i) = P(i)/T_{i+1}(P(i)) = T_i(P(i))/T_{i+1}(P(i)) \in \mathcal{P}_i$ by Lemma 2.2, so $\mathcal{F}(\Delta) \subseteq \mathcal{P}$.

It is clear that $\Delta(i)$ is the ith projective indecomposable module over $A/T_{i+1}(A)$. Since $T_{i+1}(A)$ is an idempotent ideal, the category $\mathcal{F}(\Delta(i))$ is the same over A as over the factor algebra $A/T_{i+1}(A)$, so it consists of the direct sums of copies of $\Delta(i)$. Consequently $\mathcal{F}(\Delta(i))$ is closed under direct summands and kernels of epimorphisms. If in addition, $A_A \in \mathcal{F}(\Delta)$, then by Lemma 2.2, $T_i(A)/T_{i+1}(A) \in \mathcal{F}(\Delta(i))$ for all i, so by Proposition 2.6, $\mathcal{F}(\Delta)$ is a stratifying subcategory.

The proof of the dual statement is omitted. □

Later we shall see that $\mathcal{F}(\bar{\Delta})$ is also a stratifying subcategory if A is $\bar{\Delta}$-filtered and $\mathcal{F}(\bar{\nabla})$ is a costratifying subcategory if A is Δ-filtered.

3. Stratifying pairs of subcategories

For a subcategory $\mathcal{C}$ of mod-A we use the notation

$$\mathcal{C}^{\perp} = \mathcal{C}^{\perp}_A = \{\, Y \in \text{mod-}A \mid \operatorname{Ext}^t(X,Y) = 0 \ \forall\ t > 0 \text{ and } X \in \mathcal{C} \,\}$$

and

$$^{\perp}\mathcal{C} = {}^{\perp}\mathcal{C}_A = \{\, X \in \text{mod-}A \mid \operatorname{Ext}^t(X,Y) = 0 \ \forall\ t > 0 \text{ and } Y \in \mathcal{C} \,\}.$$

It is clear that if $\mathcal{C}$ is resolving (coresolving, respectively) then in the above definitions it is enough to require that $\operatorname{Ext}^1(X,Y) = 0$.

LEMMA 3.1. *Let $\mathcal{P}$ be a stratifying subcategory. Then*

(1) $\bar{\mathcal{P}} = \mathcal{F}(\mathcal{P}_1,\dots,\mathcal{P}_{n-1}) = \mathcal{P} \cap \mathcal{F}(S(1),\dots,S(n-1))$ *and $\bar{\mathcal{P}}$ is a stratifying subcategory over $\bar{A} = A/T_n(A)$;*

(2) $\bar{\mathcal{P}}^{\perp}_{\bar{A}} = \mathcal{P}^{\perp}_A \cap \mathcal{F}(S(1),\dots,S(n-1))$.

Proof. (1) The equality $\mathcal{F}(\mathcal{P}_1,\dots,\mathcal{P}_{n-1}) = \mathcal{P} \cap \mathcal{F}(S(1),\dots,S(n-1))$ immediately follows from Lemma 2.2. Now we prove that $\bar{\mathcal{P}}$ is a stratifying subcategory over the factor algebra $\bar{A}$. As in the proof of Proposition 2.3 we get that the subclasses $\mathcal{P}_i(\mathbf{e}) \subseteq$ mod-A and $\mathcal{P}_i(\mathbf{e}') \subseteq$ mod-$\bar{A}$ are the same for $1 \le i \le n-1$ and $\mathbf{e}' = (e_1,\dots,e_{n-1})$, so $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e}')$. Furthermore, the subcategory $\mathcal{F}(\mathcal{P}_1,\dots,\mathcal{P}_{n-1})$ is the same over the two algebras, hence the required closure properties also hold. Finally, the projective modules of $\bar{A}$ are the factors $\bar{P} = P/T_n(P)$ of projective modules P over A, and Lemma 2.2 implies that $\bar{P} \in \mathcal{F}(\mathcal{P}_1,\dots,\mathcal{P}_{n-1})$.

(2) $\bar{\mathcal{P}}^{\perp}_{\bar{A}} \supseteq \mathcal{P}^{\perp}_A \cap \mathcal{F}(S(1),\dots,S(n-1))$ is clear from the fact that A is CPS-stratified. We only need to show that $\bar{\mathcal{P}}^{\perp}_{\bar{A}} \subseteq \mathcal{P}^{\perp}_A$. But for any $Y \in \bar{\mathcal{P}}^{\perp}_{\bar{A}}$ we have $Y \in \mathcal{P}^{\perp}_n$, since $Y \in \mathcal{F}(S(1),\dots,S(n-1))$, so $Y \in \mathcal{F}(\mathcal{P}_1,\dots,\mathcal{P}_{n-1},\mathcal{P}_n)^{\perp} = \mathcal{P}^{\perp}$. □

THEOREM 3.2. *If $\mathcal{P}$ is a stratifying subcategory over $(A,\mathbf{e})$ then $\mathcal{P}^{\perp}$ is costratifying. Dually, if $\mathcal{Q}$ is a costratifying subcategory over $(A,\mathbf{e})$ then $^{\perp}\mathcal{Q}$ is stratifying.*

For the proof we need two preparatory lemmas.

Lemma 3.3. *Let $\mathcal{P}$ be a stratifying subcategory. If $Y \in \mathcal{P}^{\perp}$ is such that $R_n(Y) = 0$, then $Y \in \mathcal{Q}_n(\mathbf{e})$.*

Proof. The condition $R_n(Y) = 0$ yields that $\operatorname{Hom}(S(i), Y) = 0$ for all $i < n$. By Proposition 2.13, $\Delta(i) \in \mathcal{P}_i$. Let us consider for some fixed $i < n$ the exact sequence

$$O \to U \to \Delta(i) \to S(i) \to 0.$$

This yields the long exact sequence

$$\cdots \to \operatorname{Hom}(U, Y) \to \operatorname{Ext}^1(S(i), Y) \to \operatorname{Ext}^1(\Delta(i), Y) \to \cdots$$

Here $\operatorname{Hom}(U, Y) = 0$ since $U \in \mathcal{F}(S(1), \dots, S(i))$ and $\operatorname{Hom}(S(j), Y) = 0$ for $j < n$, furthermore $\operatorname{Ext}^1(\Delta(i), Y) = 0$, because $Y \in \mathcal{P}^{\perp}$. Thus $\operatorname{Ext}^1(S(i), Y) = 0$, and this is true for all $i < n$. Now we can prove by induction that $\operatorname{Ext}^t(S(i), Y) = 0$ for all $t > 0$ and $i < n$, using the following segment of the above sequence:

$$\cdots \to \operatorname{Ext}^t(U, Y) \to \operatorname{Ext}^{t+1}(S(i), Y) \to \operatorname{Ext}^{t+1}(\Delta(i), Y) \to \cdots$$

□

Lemma 3.4. *Let $\mathcal{P}$ be a stratifying subcategory. If $Y \in \mathcal{P}^{\perp}$, then $R_n(Y) \in \mathcal{P}^{\perp}$ and $\widetilde{Y} = Y/R_n(Y) \in \mathcal{P}^{\perp}$.*

Proof. Consider the exact sequence

$$0 \to R_n(Y) \to Y \to \widetilde{Y} \to 0$$

and let us take a module $X \in \mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_{n-1})$. Then we get the induced long exact sequence

$$\cdots \to \operatorname{Hom}(X, \widetilde{Y}) \to \operatorname{Ext}^1(X, R_n(Y)) \to \operatorname{Ext}^1(X, Y) \to \cdots$$

Here $X \in \mathcal{F}(S(1), \dots, S(n-1))$ implies that $\operatorname{Hom}(X, \widetilde{Y}) = 0$, furthermore by assumption $\operatorname{Ext}^1(X, Y) = 0$, thus $\operatorname{Ext}^1(X, R_n(Y)) = 0$. On the other hand, if $X \in \mathcal{P}_n$ then $\operatorname{Ext}^1(X, R_n(Y)) = 0$ since $R_n(Y) \in \mathcal{F}(S(1), \dots, S(n-1))$. Thus $\operatorname{Ext}^1(X, R_n(Y)) = 0$ for any $X \in \mathcal{P}$. Using the fact that $\mathcal{P}$ is a resolving subcategory, we get that $R_n(Y) \in \mathcal{P}^{\perp}$. From the same long exact sequence we get now that $\operatorname{Ext}^t(X, \widetilde{Y}) = 0$ for all $t > 0$ and $X \in \mathcal{P}$, i. e. $\widetilde{Y} \in P^{\perp}$ as well. □

Proof of Theorem 3.2. Clearly, $\mathcal{Q} = \mathcal{P}^{\perp}$ is a coresolving subcategory. We only need to show that $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \dots, \mathcal{Q}_n)$, where $\mathcal{Q}_i = \mathcal{Q} \cap \mathcal{Q}_i(\mathbf{e})$. By Lemma 3.4, every element of $\mathcal{Q}$ is filtered by $\mathcal{Q} \cap \mathcal{F}(S(1), \dots, S(n-1)) \cup \{\, Y \in \mathcal{Q} \mid R_n(Y) = 0 \,\}$. By Lemma 3.1 we know that $\mathcal{Q} \cap \mathcal{F}(S(1), \dots, S(n-1)) = \bar{\mathcal{P}}^{\perp}_{\bar{A}}$ with $\bar{A} = A/T_n(A)$. Thus we may use induction on the number of simple types to get that $\mathcal{Q} \cap \mathcal{F}(S(1), \dots, S(n-1)) = \mathcal{F}(\mathcal{Q}_1, \dots, \mathcal{Q}_{n-1})$. On the other hand, by Lemma 3.3 $\{\, Y \in \mathcal{Q} \mid R_n(Y) = 0 \,\} = \mathcal{Q} \cap \mathcal{Q}_n(\mathbf{e})$, so $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \dots, \mathcal{Q}_n)$.

The dual statement can be proved along the same lines. □

Definition. A pair $(\mathcal{P},\mathcal{Q})$ of subcategories in mod-A will be called a *stratifying pair* if $\mathcal{P}$ is a stratifying subcategory and $\mathcal{Q}$ is a costratifying subcategory such that $\mathcal{P}^\perp=\mathcal{Q}$ and ${}^\perp\mathcal{Q}=\mathcal{P}$.

Note that in the literature a pair of subcategories $(\mathcal{X},\mathcal{Y})$ satisfying $\mathcal{Y}=\mathcal{X}^\perp$ and $\mathcal{X}={}^\perp\mathcal{Y}$ is called a *(hereditary) cotorsion pair*, a notion introduced by Salce. For variants of the definition of cotorsion pairs, see for example [KS] or [GT].

It is easy to find stratifying pairs over standardly stratified algebras.

Theorem 3.5. *Let* $(A,\mathbf{e})$ *be a standardly stratified algebra.*

(1) *If* A *is* $\bar\Delta$*-filtered then* $(\mathcal{F}(\bar\Delta),\mathcal{F}(\nabla))$ *is a stratifying pair, and* $\mathcal{F}(\bar\Delta)=\mathcal{P}(\mathbf{e})$.
(2) *If* A *is* Δ*-filtered then* $(\mathcal{F}(\Delta),\mathcal{F}(\bar\nabla))$ *is a stratifying pair, and* $\mathcal{F}(\bar\nabla)=\mathcal{Q}(\mathbf{e})$.

Proof. We shall prove only (1); then (2) will follow by duality.

If A is $\bar\Delta$-filtered, then from Proposition 2.13 we get that $\mathcal{F}(\nabla)$ is a costratifying subcategory. Theorem 3.1 of [ADL1] implies that $\mathcal{F}(\bar\Delta)={}^\perp\mathcal{F}(\nabla)$, hence by Theorem 3.2 we get that $\mathcal{F}(\bar\Delta)$ is a stratifying subcategory. In order to prove that $(\mathcal{F}(\bar\Delta),\mathcal{F}(\nabla))$ is a stratifying pair we still have to show that $\mathcal{F}(\nabla)=\mathcal{F}(\bar\Delta)^\perp$. We use induction on the number of simple types.

The statement clearly holds for $n=1$, since in this case $\mathcal{F}(\bar\Delta)=$ mod-A, and $\mathcal{F}(\nabla)$ is the category of injective modules. Now let $n>1$ and assume that the statement is true for the $\bar\Delta$-filtered algebra $(\bar A,\mathbf{e}')$ with $\bar A=A/T_n(A)$ and $\mathbf{e}'=(e_1,\dots,e_{n-1})$. Then $\mathcal{F}_A(\bar\Delta)^\perp\cap\mathcal{Q}(\mathbf{e}')=\mathcal{F}_{\bar A}(\bar\Delta)^\perp$ by Lemma 3.1, and the induction hypothesis implies that $\mathcal{F}_{\bar A}(\bar\Delta)^\perp=\mathcal{F}_{\bar A}(\nabla)=\mathcal{F}_A(\nabla(1),\dots,\nabla(n-1))$. On the other hand, we shall prove that every module $Y\in\mathcal{F}(\bar\Delta)^\perp\cap\mathcal{Q}_n(\mathbf{e})$ is injective, i. e. $\operatorname{Ext}^t(S(i),Y)=0$ for all $i=1,\dots,n$ and $t>0$. Indeed, the fact that $\operatorname{Ext}^t(S(i),Y)=0$ for $i<n$ and $t\ge 0$ follows from $Y\in\mathcal{Q}_n(\mathbf{e})$. Let us take the exact sequence

$$0\to U\to\bar\Delta(n)\to S(n)\to 0$$

and the induced long exact sequence

$$\cdots\to\operatorname{Ext}^t(U,Y)\to\operatorname{Ext}^{t+1}(S(n),Y)\to\operatorname{Ext}^{t+1}(\bar\Delta(n),Y)\to\cdots$$

Here $\operatorname{Ext}^t(U,Y)=0$ holds for $t\ge 0$ since $U\in\mathcal{F}(S(1),\dots,S(n-1))$, while $\operatorname{Ext}^{t+1}(\bar\Delta(n),Y)=0$ follows from $Y\in\mathcal{F}(\bar\Delta)^\perp$, so $\operatorname{Ext}^{t+1}(S(n),Y)=0$ for $t\ge 0$. This shows that $\operatorname{Ext}^t(S(i),Y)=0$ for $i=1,\dots,n$ and $t>0$, i. e. Y is injective. Thus $\mathcal{F}(\bar\Delta)^\perp\cap\mathcal{Q}_n(\mathbf{e})=\mathcal{F}(\nabla(n))$. Since $\mathcal{F}(\bar\Delta)^\perp$ is a costratifying subcategory, Lemma 2.11 implies that $\mathcal{F}(\bar\Delta)^\perp=\mathcal{F}(\nabla)$.

Finally, $\mathcal{F}(\bar\Delta)\subseteq\mathcal{P}(\mathbf{e})$ implies $\mathcal{P}(\mathbf{e})^\perp\subseteq\mathcal{F}(\bar\Delta)^\perp=\mathcal{F}(\nabla)$ and since $\mathcal{P}(\mathbf{e})^\perp$ is a costratifying subcategory, we get from Proposition 2.13 that $\mathcal{F}(\nabla)\subseteq\mathcal{P}(\mathbf{e})^\perp$. Thus $\mathcal{P}(\mathbf{e})^\perp=\mathcal{F}(\nabla)$. In this way we get $\mathcal{P}(\mathbf{e})\subseteq{}^\perp(\mathcal{P}(\mathbf{e})^\perp)={}^\perp\mathcal{F}(\nabla)=\mathcal{F}(\bar\Delta)\subseteq\mathcal{P}(\mathbf{e})$, hence $\mathcal{F}(\bar\Delta)=\mathcal{P}(\mathbf{e})$. □

Our next goal is to find stratifying pairs for arbitrary CPS-stratified algebras. The first statement of the next proposition is an immediate consequence of Theorem 3.2.

Theorem 3.6. *Suppose $(A, \mathbf{e})$ is a CPS-stratified algebra.*

(1) *If $\mathcal{P}$ is a stratifying subcategory in* mod-A *then $({}^{\perp}(\mathcal{P}^{\perp}), \mathcal{P}^{\perp})$ is a stratifying pair. Dually, if $\mathcal{Q}$ is a costratifying subcategory in* mod-A *then $({}^{\perp}\mathcal{Q}, ({}^{\perp}\mathcal{Q})^{\perp})$ is a stratifying pair. In particular, $(\mathcal{P}(\mathbf{e}), \mathcal{P}(\mathbf{e})^{\perp})$ and $({}^{\perp}\mathcal{Q}(\mathbf{e}), \mathcal{Q}(\mathbf{e}))$ are stratifying pairs.*

(2) *Let $\mathcal{M} = \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_n$, where $\mathcal{M}_i \subseteq \mathcal{P}_i(\mathbf{e})$. Then $\mathcal{Q} = \mathcal{M}^{\perp} \cap \mathcal{Q}(\mathbf{e})$ is a costratifying subcategory. Furthermore, $({}^{\perp}\mathcal{Q}, \mathcal{Q})$ form a stratifying pair in* mod-A.

Proof. Theorem 3.2 implies that the mappings $\mathcal{P} \mapsto \mathcal{P}^{\perp}$ and $\mathcal{Q} \mapsto {}^{\perp}\mathcal{Q}$ define an order reversing Galois connection between the set of stratifying and the set of costratifying subcategories. Hence the first statement of (1) follows immediately. Since $\mathcal{P}(\mathbf{e})$ is the largest stratifying subcategory, ${}^{\perp}(\mathcal{P}(\mathbf{e})^{\perp}) = \mathcal{P}(\mathbf{e})$, hence $(\mathcal{P}(\mathbf{e}), \mathcal{P}(\mathbf{e})^{\perp})$ is a stratifying pair. The dual statements follow similarly.

Let us now prove the statements in (2). Since $\mathcal{M}^{\perp}$ and $\mathcal{Q}(\mathbf{e})$ are clearly coresolving, so is their intersection.

We shall use the notation $\mathcal{Q}_i = \mathcal{Q} \cap \mathcal{Q}_i(\mathbf{e})$ and $\overline{\mathcal{M}} = \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_{n-1}$. We still need to prove that $\mathcal{Q} = \mathcal{F}(Q_1, \ldots, \mathcal{Q}_n)$; we shall use induction on n. Thus we may assume that the statement holds for $\overline{\mathcal{M}} \subseteq$ mod-$\bar{A}$ with $\bar{A} = A/T_n(A)$, i.e. $\overline{M}^{\perp} \cap \mathcal{Q}(\mathbf{e}') = \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_{n-1})$ with $\mathbf{e}' = (e_1, \ldots, e_{n-1})$.

For $Y \in \mathcal{Q}$ let us consider the short exact sequence

$$0 \to R_n(Y) \to Y \to \widetilde{Y} \to 0.$$

Lemma 2.8 implies that $R_n(Y) \in \mathcal{Q}(\mathbf{e}')$ and $\widetilde{Y} \in \mathcal{Q}_n(\mathbf{e})$. For arbitrary $X_1 \in \overline{\mathcal{M}}$ we get the long exact sequence

$$\cdots \to \operatorname{Ext}^t(X_1, \widetilde{Y}) \to \operatorname{Ext}^{t+1}(X_1, R_n(Y)) \to \operatorname{Ext}^{t+1}(X_1, Y) \to \cdots$$

Here $\operatorname{Ext}^t(X_1, \widetilde{Y}) = 0$ for $t \geq 0$, since $X_1 \in \mathcal{F}(S(1), \ldots, S(n-1))$ and $\widetilde{Y} \in \mathcal{Q}_n(\mathbf{e})$, while $\operatorname{Ext}^{t+1}(X_1, Y) = 0$ follows from $Y \in \mathcal{M}^{\perp}$. Thus the middle term is also 0, so $R_n(Y) \in \overline{M}^{\perp} \cap \mathcal{Q}(\mathbf{e}')$, i.e. $R_n(Y) \in \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_{n-1})$ by the induction hypothesis. On the other hand, for any $X_2 \in \mathcal{M}_n \subseteq \mathcal{P}_n(\mathbf{e})$ we have $\operatorname{Ext}^t(X_2, R_n(Y)) = 0$ for all $t \geq 0$, since $R_n(Y) \in \mathcal{F}(S(1), \ldots, S(n-1))$. Thus, putting together the two cases, we get that $R_n(Y) \in \mathcal{M}^{\perp}$.

Let us take now an arbitrary $X \in \mathcal{M}$. Then in the long exact sequence

$$\cdots \to \operatorname{Ext}^t(X, Y) \to \operatorname{Ext}^t(X, \widetilde{Y}) \to \operatorname{Ext}^{t+1}(X, R_n(Y)) \to \cdots$$

$\operatorname{Ext}^t(X, Y) = \operatorname{Ext}^{t+1}(X, R_n(Y)) = 0$ for all $t > 0$. Thus $\widetilde{Y} \in \mathcal{M}^{\perp} \cap \mathcal{Q}_n(\mathbf{e}) = \mathcal{Q}_n$, so $Y \in \mathcal{F}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$. This proves that $\mathcal{Q}$ is a costratifying subcategory.

It follows from Theorem 3.2 that ${}^{\perp}\mathcal{Q}$ is a stratifying subcategory. The inclusion $\mathcal{M} \subseteq {}^{\perp}(\mathcal{M}^{\perp} \cap \mathcal{Q}(\mathbf{e})) = {}^{\perp}\mathcal{Q}$ implies that $({}^{\perp}\mathcal{Q})^{\perp} \subseteq \mathcal{M}^{\perp}$. Since $({}^{\perp}\mathcal{Q})^{\perp} \subseteq \mathcal{Q}(\mathbf{e})$ clearly holds, we have $({}^{\perp}\mathcal{Q})^{\perp} \subseteq {}^{\perp}\mathcal{M} \cap \mathcal{Q}(\mathbf{e}) = \mathcal{Q}$. Finally, since the containment $({}^{\perp}\mathcal{Q})^{\perp} \supseteq \mathcal{Q}$ is obvious, we get from part (1) that $({}^{\perp}\mathcal{Q}, \mathcal{Q})$ is a stratifying pair. □

In general, not every perpendicular pair of modules $X \in \mathcal{P}(\mathbf{e})$ and $Y \in \mathcal{Q}(\mathbf{e})$ is contained in a stratifying pair of subcategories. However we have the following characterization.

Proposition 3.7. *Let $(A, \mathbf{e})$ be a CPS-stratified algebra. For arbitrary modules $X \in \mathcal{P}(\mathbf{e})$ and $Y \in \mathcal{Q}(\mathbf{e})$ the following are equivalent.*

(*i*) *There is a stratifying pair $(\mathcal{P}, \mathcal{Q})$ in* mod-A *such that $X \in \mathcal{P}$ and $Y \in \mathcal{Q}$.*
(*ii*) $X_i = T_i(X)/T_{i+1}(X) \in {}^{\perp}\{\, Y \,\}$ *for* $i = 1, \ldots, n$.
(*ii*)′ $Y_i = R_{i+1}(Y)/R_i(Y) \in \{\, X \,\}^{\perp}$ *for* $i = 1, \ldots, n$.
(*iii*) $\operatorname{Ext}^t(X_i, Y_i) = 0$ *for all* $t > 0$ *and* $i = 1, \ldots, n$.

Proof. (*i*) ⇒ (*iii*): By Lemma 2.2 and 2.8, $X_i \in \mathcal{P}_i \subseteq \mathcal{P}$ and $Y_i \in \mathcal{Q}_i \subseteq \mathcal{Q}$, so (*iii*) follows from the fact that $\mathcal{P}$ and $\mathcal{Q}$ form a cotorsion pair.

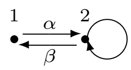
(*iii*) ⇒ (*ii*): Since $X_i \in \mathcal{P}_i(\mathbf{e})$ and $Y_j \in \mathcal{Q}_j(\mathbf{e})$ by Lemma 2.2 and 2.8, we have $\operatorname{Ext}^t(X_i, Y_j) = 0$ for all $i \neq j$ and $t \geq 0$. Together with (*iii*) this implies that $\operatorname{Ext}^t(X_i, Y) = 0$ for all $t > 0$ and $i = 1, \ldots, n$.

(*ii*) ⇒ (*i*): Take $\mathcal{M}_i = \{\, X_i \,\}$ in part (2) of Theorem 3.6. Then $Y \in \mathcal{Q}$. Furthermore $X_i \in {}^{\perp}\mathcal{Q} = \mathcal{P}$ for $i = 1 \ldots, n$, implying that $X \in \mathcal{P}$.

Finally, (*iii*) ⇒ (*ii*′) ⇒ (*i*) follows by duality. □

A similar statement can be formulated giving a condition for arbitrary subclasses of $\mathcal{P}(\mathbf{e})$ and $\mathcal{Q}(\mathbf{e})$ to be included into a stratifying pair.

It is easy to see that if $(A, \mathbf{e})$ is *quasi-hereditary* (i. e. standardly stratified with $\Delta(i) = \bar{\Delta}(i)$ for $i = 1, \ldots, n$) then $(\mathcal{P}(\mathbf{e}), \mathcal{Q}(\mathbf{e}))$ is the only stratifying pair. This follows from the fact that, by Proposition 2.13 and Theorem 3.5, $\mathcal{F}(\Delta) = \mathcal{F}(\bar{\Delta}) = \mathcal{P}(\mathbf{e})$ is the only stratifying subcategory. Non-quasi-hereditary examples with a unique stratifying pair can also be found. On the other hand the following example shows that for a general CPS-stratified algebra there may be even infinitely many different stratifying pairs of subcategories.

Example 3.8. Let $A = K\Gamma/I$, where Γ: 

and $I = (\alpha\gamma, \beta\alpha\beta, \gamma^2)$. Then

$$A_A = \begin{smallmatrix} 1 \\ 2 \\ 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 2 \\ 2 & & 1 \\ & & 2 \end{smallmatrix} \quad \text{and} \quad D({}_AA) = \begin{smallmatrix} 1 & & 2 \\ & 2 & \\ & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 2 & \\ 2 & & \\ 1 & & 2 \\ & 2 & \end{smallmatrix}.$$

Let $M = \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix}$, $N = \begin{smallmatrix} 1 & & 2 \\ & 2 & \end{smallmatrix}$, and $M_c = P(2)/(\beta\alpha - c\gamma)$ for any $0 \neq c \in K$. Then $\mathcal{M} = \{\, M, M_c \,|\, 0 \neq c \in K \,\} \subseteq \mathcal{P}_2(\mathbf{e})$ and $\mathcal{N} = \{\, N, M_c \,|\, 0 \neq c \in K \,\} \subseteq \mathcal{Q}_2(\mathbf{e})$. For any subset L of $K \setminus \{\, 0 \,\}$ take $\mathcal{M}_L = \{\, S(1), M, M_c \,|\, c \in L \,\}$. By Theorem 3.6, $\mathcal{Q}_L = \mathcal{M}_L^{\perp} \cap \mathcal{Q}(\mathbf{e})$ and $\mathcal{P}_L = {}^{\perp}\mathcal{Q}_L$ form a stratifying pair of subcategories. Easy calculation shows that $\mathcal{Q}_L \cap \mathcal{N} = \{\, N, M_d \,|\, d \in K \setminus (\{\, 0 \,\} \cup L) \,\}$, furthermore $\mathcal{P}_L \cap \mathcal{M} \subseteq {}^{\perp}(\mathcal{Q}_L \cap \mathcal{N}) \cap \mathcal{M} \subseteq \mathcal{M}_L$. Since the other inclusion is obvious, we have that $\mathcal{P}_L \cap \mathcal{M} = \mathcal{M}_L$, i. e. for each choice of $L \subseteq K \setminus \{\, 0 \,\}$ we get a different stratifying pair.

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Department of Algebra and Number Theory, Eötvös University, P.O.Box 120, H-1518 Budapest, Hungary
E-mail: agoston@cs.elte.hu

Department of Algebra, Budapest University of Technology and Economics, P.O.Box 91, H-1521 Budapest, Hungary
E-mail: lukacs@math.bme.hu