QUASI-HEREDITARY EXTENSION ALGEBRAS

István Ágoston[1,2], Vlastimil Dlab[3] and Erzsébet Lukács[1,4]

Abstract. The paper is a continuation of the authors' study of quasi-hereditary algebras whose Yoneda extension algebras (homological duals) are quasi-hereditary. The so-called standard Koszul quasi-hereditary algebras, presented in this paper, have the property that their extension algebras are always quasi-hereditary. In the natural setting of graded Koszul algebras, the converse also holds: if the extension algebra of a graded Koszul quasi-hereditary algebra is quasi-hereditary, then the algebra must be standard Koszul. This implies that the class of graded standard Koszul quasi-hereditary algebras is closed with respect to homological duality. Another immediate consequence is the fact that all algebras corresponding to the blocks of the category $\mathcal{O}$ are standard Koszul.

0. Introduction

A primary objective of the present paper is to identify a natural class of quasi-hereditary algebras which is closed with respect to homological duality and which is broad enough to accommodate applications in the representation theory of semi-simple complex Lie algebras and algebraic groups. These are related to the work of Cline, Parshall and Scott on Kazhdan–Lusztig theory which underlines the importance of those quasi-hereditary algebras whose Yoneda extension algebra is quasi-hereditary (see e.g [CPS2], [CPS3], [CPS4] and [P2]). In order to have perfect duality, the species of such an algebra must be dual to the species of its extension algebra hence these algebras are closely related to (noncommutative) Koszul algebras which have been recently extensively studied (see e.g. [BG], [BGS], [CPS4], [PS], [P1], [GM1] and [GM2]). In this way, the standard Koszul algebras of this paper, which extend the earlier notion of the solid algebras of [ADL2], are relevant for applications. Let us point out here the interesting fact that the existence of top (linear) projective resolutions for both right and left standard modules of a quasi-hereditary algebra A implies that A is a Koszul algebra. Moreover, our results show that there is a close connection between quasi-heredity and the standard Koszul property.

The main results of the present paper can be summarized in the following theorems.

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Theorem 1. *Let $(A, \mathbf{e})$ be a quasi-hereditary algebra. If both left and right standard modules have top projective resolutions, then A is a Koszul algebra (i. e. all simple modules have top projective resolutions).*

The algebras satisfying the conditions of Theorem 1 will be called *standard Koszul.*

Theorem 2. *Let $(A, \mathbf{e})$ be a standard Koszul quasi-hereditary algebra. Then the Yoneda extension algebra $(A^*, \mathbf{f})$ is also a quasi-hereditary algebra.*

Theorem 3. *Let $(A, \mathbf{e})$ be a graded Koszul quasi-hereditary algebra. Then $(A^*, \mathbf{f})$ is a quasi-hereditary algebra if and only if $(A, \mathbf{e})$ is a standard Koszul algebra. Consequently, the class of graded standard Koszul quasi-hereditary algebras is closed with respect to taking the homological dual.*

1. Standard Koszul quasi-hereditary algebras

Throughout the paper, all algebras are assumed to be basic, finite dimensional over a field K. The categories of finite dimensional right and left A-modules will be denoted by mod-A and A-mod, respectively. Given an algebra A, let $\{ e_1, e_2, \dots, e_n \}$ denote a complete set of its primitive orthogonal idempotents. Moreover $P(i) \cong e_i A$ and $S(i) \cong \operatorname{top} P(i)$ will stand for the respective indecomposable projective and simple right modules. The corresponding left modules will be denoted by $P^\circ(i)$ and $S^\circ(i)$. The direct some of all simple modules $S(i)$ will be denoted by $\hat{S}$. Thus $\hat{S} = \bigoplus_{i=1}^{n} S(i) \cong A/\operatorname{rad} A$.

Most of the concepts in this paper will depend on an order of the above set of idempotents. Their (fixed) ordering will be denoted by $\mathbf{e} = (e_1, e_2, \dots, e_n)$, and the algebra with a fixed order $\mathbf{e}$ by $(A, \mathbf{e})$. For convenience, the idempotents $e_i + e_{i+1} + \ldots + e_n$ will be denoted by ε_i, $1 \leq i \leq n$ and we put $\varepsilon_{n+1} = 0$. The *right standard module* $\Delta(i)$ (with respect to $\mathbf{e}$) is the largest quotient of $P(i)$ with no composition factor isomorphic to $S(j)$ for some $j > i$. In other terms, $\Delta(i) \cong e_i A / e_i A \varepsilon_{i+1} A$. The *left standard modules* will be denoted by $\Delta^\circ(i)$. Having fixed an order $\mathbf{e}$, we shall consider also the following two sequences of algebras: $B_i = A / A\varepsilon_{i+1} A$ and $C_i = \varepsilon_i A \varepsilon_i$; the induced order of their idempotents in both cases will be denoted by $\mathbf{e}_i$.

An idempotent ideal $I = AeA \lhd A$, generated by an idempotent element e is called a *heredity ideal* of A if $I_A \in$ mod-A is projective and $e(\operatorname{rad} A)e = 0$. Given an algebra $(A, \mathbf{e})$, denote by I_i the idempotent ideal $A\varepsilon_i A$. The algebra $(A, \mathbf{e})$ is called *quasi-hereditary* if the ideal I_i / I_{i+1} is a heredity ideal of A/I_{i+1} for all $1 \leq i \leq n$ ([CPS1], [PS1] or [DR]). This is equivalent to the requirement that A_A has a filtration with standard modules and all the standard modules are Schurian (i.e. their endomorphism algebras are division algebras).

A primitive idempotent $e \in A$ is called *neat* if for the corresponding simple module $S = \operatorname{top} eA$, the extension modules $\operatorname{Ext}^t_A(S, S) = 0$ for all $t > 0$. The algebra $(A, \mathbf{e})$ is *neat* if e_1 is neat and the centralizer algebra $(C_2, \mathbf{e}_2)$ is neat. Note that every quasi-hereditary algebra $(A, \mathbf{e})$ is neat; moreover, all neat algebras have finite global dimension (cf. [ADW]).

For the convenience of the reader, let us recall some additional definitions used freely in the present paper. For the basic properties of these concepts we refer to [ADL1] and [ADL2]. A submodule X of a module Y is said to be a *top submodule* if $\operatorname{rad} X = X \cap \operatorname{rad} Y$; in this case, we write $X \overset{t}{\subseteq} Y$. A filtration $0 = X_0 \leq X_1 \leq \cdots \leq X_k = X$ is called a *top filtration* of X if $X_i \overset{t}{\subseteq} X_{i+1}$ for all $0 < i < t$. An algebra $(A, \mathbf{e})$ is said to be *lean* if the species of the centralizer algebras $C_i = \varepsilon_i A \varepsilon_i$ are the respective restrictions of the species of A. This is equivalent to the requirement that $\varepsilon_i (\operatorname{rad} A)^2 \varepsilon_i = \varepsilon_i (\operatorname{rad} A) \varepsilon_i (\operatorname{rad} A) \varepsilon_i$ for all $1 \leq i \leq n$. Under the assumption that all standard modules $\Delta(i)$ are Schurian, $(A, \mathbf{e})$ is lean if and only if all kernels of the canonical epimorphisms $P(i) \to \Delta(i)$ and $P^\circ(i) \to \Delta^\circ(i)$ are top submodules of $\operatorname{rad} P(i)$ and $\operatorname{rad} P^\circ(i)$, respectively. A projective resolution $\cdots \xrightarrow{f_{i+1}} P_i \xrightarrow{f_i} P_{i-1} \xrightarrow{f_{i-1}} \cdots \xrightarrow{f_1} P_0 \longrightarrow X \longrightarrow 0$ is a *top resolution* of X if $\operatorname{Im} f_i \overset{t}{\subseteq} \operatorname{rad} P_{i-1}$ for every $i \geq 1$. The subcategory of right and left A-modules with minimal top projective resolutions will be denoted by $\mathcal{C}_A$ and $\mathcal{C}_A^\circ$, respectively. The vector space $\oplus_{i \geq 0} \operatorname{Ext}_A^i(\hat{S}, \hat{S})$ has an algebra structure with multiplication given by the Yoneda composition of extensions. This algebra is called the *Yoneda extension algebra* or *homological dual* of A and will be denoted by A^*. There is a natural contravariant functor $\operatorname{Ext}_A^* : \text{mod-}A \to A^*\text{-mod}$ defined by $\operatorname{Ext}_A^*(X) = \oplus_{t \geq 0} \operatorname{Ext}_A^t(X, \hat{S})$ for every $X \in \text{mod-}A$. Given $(A, \mathbf{e})$, the idempotent elements $f_i = \operatorname{id}_{S(i)} \in A^*$ define an "opposite" order $\mathbf{f} = (f_n, f_{n-1}, \dots, f_1)$. Similarly to the definition of idempotents ε_i, write $\varphi_i = f_1 + f_2 + \cdots + f_i$ for $1 \leq i \leq n$ and $\varphi_0 = 0$.

Definition 1.1. An algebra A is *Koszul* if all simple modules $S(i)$ belong to the subcategory $\mathcal{C}_A$.

Koszul algebras are characterized by the fact that $\operatorname{Ext}_A^t(\hat{S}, \hat{S}) = \big(\operatorname{Ext}_A^1(\hat{S}, \hat{S})\big)^t$ for every $t \geq 1$; in particular, this is equivalent to the fact that the species of A^* is the dual of the species of A (cf. Theorem 2.10 of [ADL2]).

Definition 1.2. An algebra $(A, \mathbf{e})$ is *recursively Koszul* (with respect to the given order) if the centralizer algebras $C_i = \varepsilon_i A \varepsilon_i$ are Koszul for every $1 \leq i \leq n$.

Definition 1.3. An algebra $(A, \mathbf{e})$ is *standard Koszul* if for every $1 \leq i \leq n$ the right and left standard modules $\Delta(i)$ and $\Delta^\circ(i)$ belong to $\mathcal{C}_A$ and $\mathcal{C}_A^\circ$, respectively.

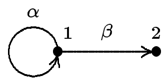
The following theorem shows that these two notions are closely related.

Theorem 1.4. *For a quasi-hereditary algebra $(A, \mathbf{e})$ the following two statements are equivalent:*

(i) $(A, \mathbf{e})$ is recursively Koszul and lean;
(ii) $(A, \mathbf{e})$ is standard Koszul.

Remark 1.5. The following two examples show that without the assumption that A is quasi-hereditary, Theorem 1.4 does not hold.

Consider the algebra $A = KQ/I$, where the quiver Q is



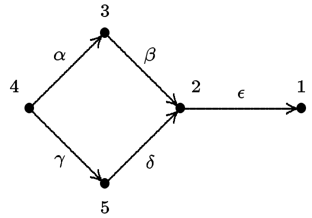
and $I = \langle \alpha^2 \rangle$; thus, the right regular representation of A is

$$A_A = \begin{smallmatrix} & 1 & \\ & 1 & 2 \\ 2 & & \end{smallmatrix} \oplus 2 .$$

This algebra is recursively Koszul and lean, but $\Delta(1) \notin \mathcal{C}_A$. Observe, that the algebra is standardly stratified (cf. [ADL3]), but not quasi-hereditary.

On the other hand, the algebra $K[x]/(x^3)$ is a standard Koszul algebra which is not Koszul.

Moreover, the following example shows that a mere assumption of the algebra to be neat is also not sufficient for the conclusions of Theorem 1.4. Let $A = KQ/I$, where Q is the quiver



and $I = \langle \beta\epsilon, \alpha\beta - \gamma\delta \rangle$. Thus, the right regular representation of A is

$$A_A = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 4 & \\ 3 & & 5 \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 2 \\ 1 \end{smallmatrix} .$$

It turns out that $(A, \mathbf{e})$ is a recursively Koszul, neat and lean algebra which is neither standard Koszul nor quasi-hereditary.

To prove Theorem 1.4, we need a series of preparatory lemmas.

Lemma 1.6. *Let $\varepsilon \in A$ be an idempotent element, and $X, Y \in \text{mod-}A$ with $X \subseteq \operatorname{rad} Y$, $X = X\varepsilon A$ and $Y = Y\varepsilon A$.*

(i) If $X \overset{t}{\subseteq} \operatorname{rad} Y$, then $X\varepsilon \overset{t}{\subseteq} \operatorname{rad}(Y\varepsilon)$.

(ii) If $\varepsilon(\operatorname{rad}^2 A)\varepsilon \subseteq \varepsilon(\operatorname{rad} A)\varepsilon(\operatorname{rad} A)\varepsilon$ and $X\varepsilon \overset{t}{\subseteq} \operatorname{rad}(Y\varepsilon)$, then $X \overset{t}{\subseteq} \operatorname{rad} Y$.

Proof. Let us denote the radical of A by J. To prove (i), observe that

$$X\varepsilon \cap Y\varepsilon(\varepsilon J \varepsilon)^2 \subseteq X\varepsilon \cap YJ^2\varepsilon = (X \cap YJ^2)\varepsilon = XJ\varepsilon = X\varepsilon(\varepsilon J\varepsilon),$$

since $X \overset{t}{\subseteq} \operatorname{rad} Y$ and $X = X\varepsilon A$.

To prove (ii), we have to show that $X \cap YJ^2 \subseteq XJ$.

First, making use of the assumptions $Y = Y\varepsilon A$, $\varepsilon J^2\varepsilon \subseteq \varepsilon J\varepsilon J\varepsilon$ and $X\varepsilon \overset{t}{\subseteq} \operatorname{rad}(Y\varepsilon)$, we get the following sequence of inclusions :

$$(X \cap YJ^2)\varepsilon \subseteq X\varepsilon \cap YJ^2\varepsilon = X\varepsilon \cap Y\varepsilon J^2\varepsilon \subseteq X\varepsilon \cap Y\varepsilon J\varepsilon J\varepsilon = \\ = X\varepsilon \cap Y\varepsilon(\varepsilon J\varepsilon)^2 \subseteq X\varepsilon(\varepsilon J\varepsilon) \subseteq XJ.$$

Furthermore, the following inclusions obviously hold:

$$(X \cap YJ^2)(1-\varepsilon) \subseteq X(1-\varepsilon) = X\varepsilon A(1-\varepsilon) \subseteq XJ.$$

Consequently, $X \cap YJ^2 \subseteq XJ$, i. e. $X \overset{t}{\subseteq} \operatorname{rad} Y$, as required. □

LEMMA 1.7. *Suppose $\varepsilon \in A$ is an idempotent element such that $\varepsilon(\operatorname{rad}^2 A)\varepsilon \subseteq \varepsilon(\operatorname{rad} A)\varepsilon(\operatorname{rad} A)\varepsilon$. (This is satisfied, in particular, when $(A, \mathbf{e})$ is lean and $\varepsilon = \varepsilon_i$ for some $1 \le i \le n$.) Let $X \in$ mod-A be such that $\operatorname{Ext}_A^t(X, \operatorname{top}((1-\varepsilon)A)) = 0$ for $t \ge 0$. Then $X \in \mathcal{C}_A$ if and only if $X\varepsilon \in \mathcal{C}_{\varepsilon A\varepsilon}$.*

Proof. Let us consider the minimal projective resolution of X over A

$$\cdots \to P_t \to \cdots \to P_1 \to P_0 \to X \to 0$$

and the corresponding exact sequence over $\varepsilon A\varepsilon$, obtained by applying the exact functor $\operatorname{Hom}_A(\varepsilon A, -)$

$$\cdots \to P_t\varepsilon \to \cdots \to P_1\varepsilon \to P_0\varepsilon \to X\varepsilon \to 0.$$

The assumptions imply that $P_t = P_t\varepsilon A$ for every $t \ge 0$, so all modules $P_t\varepsilon \in$ mod-$\varepsilon A\varepsilon$ are projective, and the second sequence is also a minimal projective resolution. Now, applying Lemma 1.6 (i) and (ii), the statement follows. □

DEFINITION 1.8. Let S be a simple module in mod-A. We say that a module $X \in$ mod-A is *S-Koszul*, if $\operatorname{Ext}_A^t(X, S) \subseteq \operatorname{Ext}_A^1(\hat{S}, S) \cdot \operatorname{Ext}_A^{t-1}(X, \hat{S})$ holds for every $t \ge 1$, where $\hat{S} = A/\operatorname{rad} A$.

Let us point out that A is S-Koszul if and only if, for every $t \ge 1$, the t-th syzygy

$$0 \to \Omega_t \to P_{t-1} \to \cdots \to P_1 \to P_0 \to X \to 0$$

of a minimal projective resolution of X satisfies the following condition: the induced morphism from the S-homogeneous summand of $\Omega_t/\operatorname{rad}\Omega_t$ into $\operatorname{rad} P_{t-1}/\operatorname{rad}^2 P_{t-1}$ is a monomorphism. Observe that this implies that whenever X is S-Koszul, then all syzygies of X are also S-Koszul. Let us also note that any module $X \in \mathcal{C}_A$ is S-Koszul for every simple module S.

LEMMA 1.9. *Suppose that $\operatorname{Ext}_A^t(S(1), S(1)) = 0$ for $t \ge 1$ (i. e. e_1 is a neat idempotent). If $X \in$ mod-A is $S(1)$-Koszul, then $X\varepsilon_2 A$ is also $S(1)$-Koszul.*

Proof. The short exact sequence

$$0 \to X\varepsilon_2 A \stackrel{\iota}{\longrightarrow} X \to \oplus S(1) \to 0$$

induces the long exact sequence

$$\cdots \to \operatorname{Ext}^t_A(\oplus S(1), S(1)) \to \operatorname{Ext}^t_A(X, S(1)) \to \operatorname{Ext}^t_A(X\varepsilon_2 A, S(1)) \to \\ \to \operatorname{Ext}^{t+1}_A(\oplus S(1), S(1)) \to \cdots,$$

where the first and the last terms are zero for every $t \geq 1$. Thus the morphism $\operatorname{Ext}^t_A(X, S(1)) \to \operatorname{Ext}^t_A(X\varepsilon_2 A, S(1))$ is an isomorphism and we have

$$\operatorname{Ext}^t_A(X\varepsilon_2 A, S(1)) = \operatorname{Ext}^t_A(X, S(1)) \cdot \iota \subseteq \operatorname{Ext}^1_A(\hat{S}, S(1)) \cdot \operatorname{Ext}^{t-1}_A(X, \hat{S}) \cdot \iota \subseteq \\ \subseteq \operatorname{Ext}^1_A(\hat{S}, S(1)) \cdot \operatorname{Ext}^{t-1}_A(X\varepsilon_2 A, \hat{S}).$$

Hence $X\varepsilon_2 A$ is $S(1)$-Koszul. □

Now, given an algebra $(A, \mathbf{e})$, let us define the following subclass $\mathcal{K}$ of mod-A:

$$\mathcal{K} = \left\{ X \in \text{mod-}A \mid X \text{ is } S(1)\text{-Koszul}, \ X\varepsilon_2 \in \mathcal{C}_{C_2} \text{ and } X\varepsilon_2 A \stackrel{t}{\subseteq} X \right\}.$$

Furthermore, let us introduce the following correspondence $\mu : \text{mod-}A \to \text{mod-}A$. For $X \in \text{mod-}A$, define

$$\mu(X) = \begin{cases} X\varepsilon_2 A & \text{if } X \neq X\varepsilon_2 A, \\ \Omega_1(X) & \text{if } X = X\varepsilon_2 A, \end{cases}$$

where $\Omega_1(X)$ is the first syzygy of X.

LEMMA 1.10. *Let $(A, \mathbf{e})$ be a lean and neat algebra with $S(1) \in \mathcal{C}_A$. Let $\mathcal{K}$ and μ be defined as above. Then:*

(i) $\mu(\mathcal{K}) \subseteq \mathcal{K}$;
(ii) for every $X \in \text{mod-}A$, there exists a positive integer k such that $\mu^k(X) = 0$;
(iii) $\mathcal{K} \subseteq \mathcal{C}_A$.

Proof. To prove (i) suppose that $X \in \mathcal{K}$. If $X \neq X\varepsilon_2 A$ then $\mu(X) = X\varepsilon_2 A$ is $S(1)$-Koszul by Lemma 1.9 and the other two conditions in the definition of $\mathcal{K}$ obviously hold for $\mu(X)$. So $\mu(X) \in \mathcal{K}$. Let us now suppose that $X = X\varepsilon_2 A$. Then $\mu(X) = \Omega_1(X)$ is $S(1)$-Koszul according to the remark following the definition of S-Koszul modules. Furthermore, the exact sequence

$$0 \to \Omega \to P \to X \to 0, \tag{1}$$

with $\Omega = \Omega_1(X)$ and $P \to X$ the projective cover of X, yields the exact sequence

$$0 \to \Omega\varepsilon_2 \to P\varepsilon_2 \to X\varepsilon_2 \to 0$$

in mod-C_2. The assumption $X = X\varepsilon_2 A$ implies that $P\varepsilon_2 \to X\varepsilon_2$ is a projective cover of $X\varepsilon_2 \in \mathcal{C}_{C_2}$. Thus $\Omega\varepsilon_2 \in \mathcal{C}_{C_2}$ and $\Omega\varepsilon_2 \overset{t}{\subseteq} \mathrm{rad}(P\varepsilon_2)$. Now, Lemma 1.6 (ii) yields that $\Omega\varepsilon_2 A \overset{t}{\subseteq} \mathrm{rad}\, P$, and consequently, $\Omega\varepsilon_2 A \overset{t}{\subseteq} \Omega$. Hence $\mu(X) = \Omega \in \mathcal{K}$.

To prove (ii), let us attach to every module $X \in$ mod-A the following two nonnegative integers:

$$e_X = \dim_K \bigoplus_{t\geq 0} \mathrm{Ext}^t_A(X, S(1)) \text{ and } p_X = pd\, X.$$

Note that both of these numbers must be finite since neat algebras have finite global dimension. Observe also that $(e_X, p_X) = (0,0)$ if and only if $X = X\varepsilon_2 A$ and X is projective. In this case $\mu(X) = 0$. Thus to prove (ii), it will be enough to show that $(e_{\mu(X)}, p_{\mu(X)}) < (e_X, p_X)$ in the lexicographical ordering, whenever $(e_X, p_X) \neq (0,0)$.

In the case when $X = X\varepsilon_2 A$ and X is not projective, we have $\mu(X) = \Omega_1(X)$, and then clearly $e_{\mu(X)} = e_X$ and $p_{\mu(X)} < p_X$. On the other hand, when $X \neq X\varepsilon_2 A$, then $\mu(X) = X\varepsilon_2 A$. In this case the long exact sequence in the proof of Lemma 1.9 yields an isomorphism $\mathrm{Ext}^t_A(X\varepsilon_2 A, S(1)) \cong \mathrm{Ext}^t_A(X, S(1))$ for every $t \geq 1$. Since $\mathrm{Hom}(X\varepsilon_2 A, S(1)) = 0$, while $\mathrm{Hom}(X, S(1)) \neq 0$, it turns out that $e_{\mu(X)} < e_X$. This completes the proof of (ii).

Finally, in order to prove (iii), i.e. that every module $X \in \mathcal{K}$ belongs to $\mathcal{C}_A$, we proceed by induction on the smallest number k for which $\mu^k(X) = 0$. Note that such a number k exists for all $X \in \mathcal{K}$ by (ii). The statement is clearly true for $k = 0$, i.e. for $X = 0$.

Consider first the case when $X \neq X\varepsilon_2 A$. Then we have an exact sequence

$$0 \longrightarrow X\varepsilon_2 A \longrightarrow X \longrightarrow \oplus S(1) \longrightarrow 0$$

with $X\varepsilon_2 A \overset{t}{\subseteq} X$. Here, $S(1) \in \mathcal{C}_A$ by assumption and $X\varepsilon_2 A = \mu(X) \in \mathcal{K}$ is in $\mathcal{C}_A$ by the induction hypothesis. Thus, $X \in \mathcal{C}_A$, as well (cf. Lemma 2.4 of [ADL2]).

In the case when $X = X\varepsilon_2 A$ and $X \neq 0$, consider once more the exact sequence (1). Here $\Omega = \mu(X) \in \mathcal{K}$, so by the induction hypothesis, $\Omega \in \mathcal{C}_A$. As in the proof of (i), $\Omega\varepsilon_2 A \overset{t}{\subseteq} \mathrm{rad}\, P$. Furthermore, since X is $S(1)$-Koszul, and since the neatness of A implies that $\Omega/\Omega\varepsilon_2 A$ is semisimple, we get $\Omega/\Omega\varepsilon_2 A \overset{t}{\subseteq} \mathrm{rad}\, P/\Omega\varepsilon_2 A$, and thus $\Omega \overset{t}{\subseteq} \mathrm{rad}\, P$ (cf. Lemma 1.1.(c) of [ADL1]). This implies that $X \in \mathcal{C}_A$, finishing the proof of Lemma 1.10. □

Since by an earlier remark, all modules in $\mathcal{C}$ are $S(1)$-Koszul, the previous result shows that for neat and lean algebras with $S(1) \in \mathcal{C}_A$, the class $\mathcal{K}$ can be defined as

$$\mathcal{K} = \left\{ X \in \text{mod-}A \,|\, X \in \mathcal{C}_A, X\varepsilon_2 \in \mathcal{C}_{C_2} \text{ and } X\varepsilon_2 A \overset{t}{\subseteq} X \right\}.$$

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. $(i) \Rightarrow (ii)$: Assume that $(A, \mathbf{e})$ is lean and that all the centralizer algebras $C_i = \varepsilon_i A \varepsilon_i$ $(1 \le i \le n)$ are Koszul. Observe that $\Delta(i)\varepsilon_i = S(i) \in$ mod-C_i follows from the Schurian property of the standard modules. Furthermore, the quasi-heredity of A yields that $\mathrm{Ext}_A^t(\Delta(i), S(j)) = 0$ for every $j < i$ and every $t \ge 0$. Since C_i is Koszul, $S(i) \in \mathcal{C}_{C_i}$; thus, Lemma 1.7 implies that $\Delta(i) \in \mathcal{C}_A$. Since the properties of being quasi-hereditary, lean and Koszul are all two-sided concepts, we get $\Delta^\circ(i) \in \mathcal{C}_A^\circ$, as well.

$(ii) \Rightarrow (i)$: Suppose that A is standard Koszul, i.e. $\Delta(i) \in \mathcal{C}_A$ and $\Delta^\circ(i) \in \mathcal{C}_A^\circ$ for $1 \le i \le n$; then Theorem 2.1 of [ADL1] yields that A is lean.

The fact that A is recursively Koszul will be proved by induction on the number of simple modules over A. Observe that $(C_2, \mathbf{e}_2)$ is quasi-hereditary and that the right and left standard modules of $(C_2, \mathbf{e}_2)$ are of the form $\Delta(i)\varepsilon_2$ and $\varepsilon_2\Delta^\circ(i)$, respectively. Hence, by Lemma 1.7, they belong to $\mathcal{C}_{C_2}$ and $\mathcal{C}_{C_2}^\circ$, respectively. Thus, by the induction hypothesis, the algebras C_i $(2 \le i \le n)$ are Koszul. So we need only to show that A is Koszul.

In view of Lemma 1.10 *(iii)*, it suffices to show that $\hat{S}_A \in \mathcal{K}$. Note that $S(1) = \Delta(1) \in \mathcal{C}_A$ holds by assumption. Furthermore, $\hat{S}\varepsilon_2 \in \mathcal{C}_{C_2}$ because the algebra C_2 is Koszul. It is also clear that $\hat{S}\varepsilon_2 A \overset{t}{\subseteq} \hat{S}$. Finally, we are going to show that $\hat{S}$ is $S(1)$-Koszul. Since $S^\circ(1) = \Delta^\circ(1) \in \mathcal{C}_A^\circ$, Corollary 2.7 of [ADL2] assures that each element of the dual $\mathrm{Ext}_A^t(S^\circ(1), \hat{S}^\circ)$ of $\mathrm{Ext}_A^t(\hat{S}, S(1))$ belongs to $\mathrm{Ext}_A^1(\hat{S}^\circ, \hat{S}^\circ)^{t-1} \cdot \mathrm{Ext}_A^1(S^\circ(1), \hat{S}^\circ)$, and thus

$$\mathrm{Ext}_A^t(\hat{S}, S(1)) \subseteq \mathrm{Ext}_A^1(\hat{S}, S(1)) \cdot \mathrm{Ext}_A^1(\hat{S}, \hat{S})^{t-1} \subseteq \mathrm{Ext}_A^1(\hat{S}, S(1)) \cdot \mathrm{Ext}_A^{t-1}(\hat{S}, \hat{S}).$$

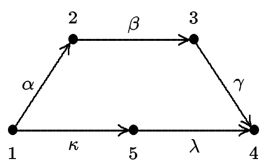
Now, an application of Lemma 1.10 (iii) yields $\hat{S} \in \mathcal{C}$, i.e. A is Koszul, as required. □

The following statement shows that the standard Koszul property of a quasi-hereditary algebra $(A, \mathbf{e})$ is carried over to the centralizer algebras $(C_i, \mathbf{e}_i)$.

PROPOSITION 1.11. *If $(A, \mathbf{e})$ is a standard Koszul quasi-hereditary algebra, then so is $(C_i, \mathbf{e}_i)$ for every $1 \le i \le n$.*

Proof. Both properties of being quasi-hereditary and lean are clearly inherited by $(C_i, \mathbf{e}_i)$ and thus $(C_i, \mathbf{e}_i)$ is a standard Koszul algebra by Theorem 1.4. □

EXAMPLE 1.12. In contrast to Proposition 1.11, the standard Koszul property of $(A, \mathbf{e})$ is not necessarily inherited by the factor algebras $(B_i, \mathbf{e}_i)$ for $1 \le i \le n-1$. To see this, consider the following quasi-hereditary algebra (Example 6.7 from [ADL2]). Let $A = KQ/I$, where Q is the quiver

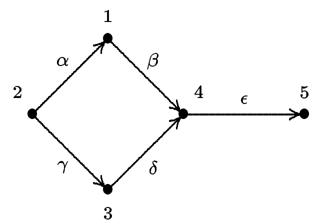


and $I = \langle \alpha\beta\gamma - \kappa\lambda \rangle$. Thus, the right regular representation of A is

$$A_A = \begin{smallmatrix} & 1 & \\ 2 & & \\ 3 & & 5 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4 \oplus \begin{smallmatrix} 5 \\ 4 \end{smallmatrix}.$$

Here, $\Delta(i) \in \mathcal{C}_A, \Delta(i)^\circ \in \mathcal{C}_A^\circ$ for $1 \leq i \leq n$, but B_4 is not Koszul since $S(1)$ does not have a top projective resolution over B_4. In Section 2, we shall give a necessary and sufficient condition for the factor algebras $(B_i, \mathbf{e}_i)$ to be standard Koszul (cf. Proposition 2.8).

EXAMPLE 1.13. The following simple example of a lean quasi-hereditary algebra $(A, \mathbf{e})$ illustrates the fact that even if all B_i, $1 \leq t \leq n$, are Koszul algebras, $(A, \mathbf{e})$ does not have to be recursively Koszul (and thus standard Koszul). Let $A = KQ/I$, where Q is the quiver



and $I = \langle \beta\epsilon, \alpha\beta - \gamma\delta \rangle$. Thus, the right regular representation of A is

$$A_A = \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus 5\,.$$

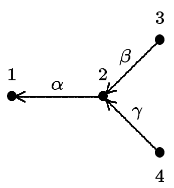
Observe that $\Delta(2) \notin \mathcal{C}$.

Finally, we show that standard Koszul quasi-hereditary algebras generalize the concept of solid algebras, defined in [ADL2]. Let us first recall the (structural) definition of a *solid* algebra $(A, \mathbf{e})$. Denoting by $U(i)$ and $V(i)$ the kernels of the canonical epimorphisms $\Delta(i) \to S(i)$ and $P(i) \to \Delta(i)$, respectively, we call an algebra $(A, \mathbf{e})$ *solid* if, for all $1 \leq i \leq n$, the multiplicity $[\Delta(i) : S(i)] = 1$ (i.e. $\Delta(i)$ is Schurian), $V(i)$ is a top submodule of $\operatorname{rad} P(i)$, $U(i)$ has a top filtration by $S(j)'s$ and $\Delta(j)'s$ (for $j < i$) and $V(i)$ has a top filtration by $\Delta(j)'s$ and $P(j)'s$ (for $j > i$). Since the centralizer algebras $(C_i, \mathbf{e}_i)$ of a solid algebra $(A, \mathbf{e})$ are solid, and since, moreover, solid algebras are lean Koszul quasi-hereditary algebras, Theorem 1.4 implies the following statement.

COROLLARY 1.14. *If* $(A, \mathbf{e})$ *is a solid algebra, then* $(A, \mathbf{e})$ *is a standard Koszul quasi-hereditary algebra.*

Note that easy examples show that not every standard Koszul algebra is solid. In particular, as the next example shows, the concept of a solid algebra is one-sided. On the other hand, it is clear that $(A, \mathbf{e})$ is standard Koszul if and only if $(A^{opp}, \mathbf{e})$ is standard Koszul.

EXAMPLE 1.15. Let us consider the algebra $A = KQ/I$, where the quiver Q is



and $I = \langle \beta\alpha \rangle$.

Thus, the right regular representation

$$A_A = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 2 \\ 1 \end{smallmatrix}.$$

shows immediately that this algebra is solid. On the other hand, the left regular representation

$$_AA = \begin{smallmatrix} 1 \\ 2 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \end{smallmatrix} \oplus 3 \oplus 4 .$$

of A shows that A^{opp} is not solid.

2. Homological dual of standard Koszul quasi-hereditary algebras

In this section we investigate the Yoneda extension algebra A^* of a standard Koszul algebra A. In particular, we establish a sufficient condition for an algebra A to have a quasi-hereditary extension algebra (with respect to the opposite order $\mathbf{f}$). As a consequence, we get that the extension algebra of a standard Koszul quasi-hereditary algebra A is quasi-hereditary. In the final Section 3, the above condition is shown to be both necessary and sufficient in the case of graded Koszul algebras.

THEOREM 2.1. *If $(A, \mathbf{e})$ is a recursively Koszul, lean and neat algebra, then the extension algebra $(A^*, \mathbf{f})$ of A is a quasi-hereditary algebra.*

To prove the quasi-heredity of the extension algebra, we shall use the following lemma. Recall that we consider the (opposite) order $\mathbf{f} = (f_n, f_{n-1}, \dots, f_1)$ of the orthogonal primitive idempotents $f_i = \mathrm{id}_{S(i)}$ in the extension algebra A^*.

LEMMA 2.2. *Let $(A, \mathbf{e})$ be a lean and neat algebra with $S(1) \in \mathcal{C}_A$ and let X belong to $\mathcal{K} = \left\{ X \in \text{mod-}A \,|\, X \in \mathcal{C}_A, X\varepsilon_2 \in \mathcal{C}_{C_2} \text{ and } X\varepsilon_2 A \overset{t}{\subseteq} X \right\}$. Then*

(*i*) *$A^* f_1 \operatorname{Ext}^*_A(X)$, the trace of the first projective left module in $\operatorname{Ext}^*(X) \in A^*\text{-mod}$ is projective.*

(*ii*) $\dim_K \operatorname{Ext}^*_A(X) - \dim_K A^* f_1 \operatorname{Ext}^*_A(X) = \dim_K \operatorname{Ext}^*_{C_2}(X\varepsilon_2)$.

In the proof of this lemma, we shall repeatedly use the following statements (Lemma 3.2 and Lemma 3.3 of [ADL2]) which we quote here without proof.

LEMMA 2.3. *Let $0 \to U \to V \to W \to 0$ be an exact sequence of modules in* mod-A.

(*i*) *If $U \overset{t}{\subseteq} V$ and $U \in \mathcal{C}_A$, then the induced sequence*

$$0 \to \operatorname{Ext}^*_A(W) \to \operatorname{Ext}^*_A(V) \to \operatorname{Ext}^*_A(U) \to 0,$$

is also exact in A^*-mod. *If, in addition* $W \in \mathcal{C}_A$, *then* $\operatorname{Ext}^*_A(W) \overset{t}{\subseteq} \operatorname{Ext}^*_A(V)$.

(*ii*) *If* $U \overset{t}{\subseteq} \operatorname{rad} V$ *and* $V \in \mathcal{C}$, *then the induced sequence*

$$0 \to \operatorname{Ext}^*_A(U) \to \operatorname{Ext}^*_A(W) \to \operatorname{Ext}^*_A(V) \to 0$$

is also exact in A^*-mod. *If, in addition* U *and* W *also belong to* $\mathcal{C}_A$, *then* $\operatorname{Ext}^*_A(U) \overset{t}{\subseteq} \operatorname{rad} \operatorname{Ext}^*_A(W)$.

Proof of Lemma 2.2. We shall prove the two statements of Lemma 2.2 simultaneously, using induction on the smallest number k such that $\mu^k(X) = 0$. Recall that such k exists in view of Lemma 1.10 (*ii*).

The statements clearly hold when $k = 0$, i.e. when $X = 0$.

Consider first the case when $X \neq X\varepsilon_2 A$. Then $\mu(X) = X\varepsilon_2 A$, and the statements hold for $X\varepsilon_2 A$ by the induction hypothesis.

Since $X\varepsilon_2 A \overset{t}{\subseteq} X$ and $X\varepsilon_2 A \in \mathcal{K} \subseteq \mathcal{C}_A$, the exact sequence in mod-A

$$0 \to X\varepsilon_2 A \to X \to \oplus S(1) \to 0$$

induces, by Lemma 2.3 (*i*) an exact sequence

$$0 \to \operatorname{Ext}^*_A(\oplus S(1)) \to \operatorname{Ext}^*_A(X) \to \operatorname{Ext}^*_A(X\varepsilon_2 A) \to 0.$$

in A^*-mod. Here, $\operatorname{Ext}^*_A(\oplus S(1)) \cong \oplus P^\circ_{A^*}(1) \cong \oplus A^* f_1$ is embedded into $A^* f_1 \operatorname{Ext}^*_A(X)$. Thus we also get the following exact sequence:

$$0 \to \operatorname{Ext}^*_A(\oplus S(1)) \to A^* f_1 \operatorname{Ext}^*_A(X) \to A^* f_1 \operatorname{Ext}^*_A(X\varepsilon_2 A) \to 0.$$

The first term $\operatorname{Ext}^*_A(\oplus S(1))$ of this sequence is clearly projective. Since the last term $A^* f_1 \operatorname{Ext}^*_A(X\varepsilon_2 A)$ is, by the induction hypothesis, also projective, the middle term is projective, as well, and (*i*) is proved. Furthermore, the two sequences over A^* yield that $\dim_K \operatorname{Ext}^*_A(X) - \dim_K A^* f_1 \operatorname{Ext}^*_A(X) = \dim_K \operatorname{Ext}^*_A(X\varepsilon_2 A) - \dim_K A^* f_1 \operatorname{Ext}^*_A(X\varepsilon_2 A)$ and by the induction hypothesis, we get that this is equal to $\dim_K \operatorname{Ext}^*_{C_2}(X\varepsilon_2)$, proving (*ii*).

Now, we turn to the case when $0 \neq X = X\varepsilon_2 A$; then $\mu(X) = \Omega$, where $0 \to \Omega \to P \to X \to 0$ with a projective cover $P \to X$. Since $X \in \mathcal{K} \subseteq \mathcal{C}_A$, clearly $\Omega \overset{t}{\subseteq} \operatorname{rad} P$. By Lemma 2.3.(*ii*), we get the following exact sequence in A^*-mod:

$$0 \to \operatorname{Ext}^*_A(\Omega) \to \operatorname{Ext}^*_A(X) \to \operatorname{Ext}^*_A(P) \to 0. \tag{1}$$

Here again $\operatorname{Ext}^*_A(\Omega) \overset{t}{\subseteq} \operatorname{rad} \operatorname{Ext}^*_A(X)$. Note that $\operatorname{Ext}^*_A(P)$ is semisimple and since $X = X\varepsilon_2 A$, it turns out that $A^* f_1 \operatorname{Ext}^*_A(P) = 0$. Consequently, $A^* f_1 \operatorname{Ext}^*_A(\Omega) = A^* f_1 \operatorname{Ext}^*_A(X)$ and thus, using the induction hypothesis for $\mu(X) = \Omega$, we get that $A^* f_1 \operatorname{Ext}^*_A(X)$ is projective, and (*i*) follows.

In order to prove (*ii*) for this case, consider the following exact sequence:

$$0 \to \Omega\varepsilon_2 \to P\varepsilon_2 \to X\varepsilon_2 \to 0.$$

Since $X = X\varepsilon_2 A$, the module $P\varepsilon_2$ is the projective cover of $X\varepsilon_2 \in$ mod-C_2; moreover, the condition $X\varepsilon_2 \in \mathcal{C}_{C_2}$ implies that $\Omega\varepsilon_2 \overset{t}{\subseteq} \mathrm{rad}(P\varepsilon_2)$ in mod-C_2. Hence, referring again to Lemma 2.3 (ii), we get the following exact sequence in C_2^*-mod:

$$0 \to \mathrm{Ext}^*_{C_2}(\Omega\varepsilon_2) \to \mathrm{Ext}^*_{C_2}(X\varepsilon_2) \to \mathrm{Ext}^*_{C_2}(P\varepsilon_2) \to 0. \tag{2}$$

Now, using the exact sequences (1) and (2), the induction hypothesis applied to $\mu(X) = \Omega$, the equality $A^* f_1 \mathrm{Ext}^*_A(\Omega) = A^* f_1 \mathrm{Ext}^*_A(X)$, and the fact that $P = P\varepsilon_2 A$, we get the following sequence of equalities:

$$\begin{aligned}
\dim_K \mathrm{Ext}^*_A(X) - \dim_K A^* f_1 \mathrm{Ext}^*_A(X) =& \\
&= \dim_K \mathrm{Ext}^*_A(\Omega) + \dim_K \mathrm{Ext}^*_A(P) - \dim_K A^* f_1 \mathrm{Ext}^*_A(\Omega) = \\
&= \dim_K \mathrm{Ext}^*_{C_2}(\Omega\varepsilon_2) + \dim_K \mathrm{Ext}^*_A(P) = \\
&= \dim_K \mathrm{Ext}^*_{C_2}(X\varepsilon_2) - \dim_K \mathrm{Ext}^*_{C_2}(P\varepsilon_2) + \dim_K \mathrm{Ext}^*_A(P) = \\
&= \dim_K \mathrm{Ext}^*_{C_2}(X\varepsilon_2).
\end{aligned}$$

This gives (ii), completing the proof of Lemma 2.2. □

Proof of Theorem 2.1. We shall proceed by induction on the number n of simple modules. For $n = 1$, both A and A^* are division algebras and the statement is trivial. Thus, let $n > 1$.

First, in view of the fact that both algebras A and C_2 are Koszul, we can apply Proposition 2.2 (i) to $X = \hat{S} = S(1) \oplus \cdots \oplus S(n)$. Observe that ${}_{A^*}A^* = \mathrm{Ext}^*_A(X)$ and conclude that the left A^*-module $A^* f_1 A^*$ is projective. Furthermore, since A is neat, $\mathrm{Ext}^t_A(S(1), S(1)) = 0$ for all $t > 0$, and thus the endomorphism algebra $f_1 A^* f_1 = \mathrm{Hom}_A(S(1), S(1))$ is a division algebra. Hence, $A^* f_1 A^*$ is a heredity ideal in A^*.

In order to complete the proof by induction, it is sufficient to show that $A^*/A^* f_1 A^* \cong C_2{}^*$. Evidently, for any algebra $(A, \mathbf{e})$ there is an algebra homomorphism $\Psi : A^* \to C_2{}^*$ induced by the exact functor $\mathrm{Hom}_A(\varepsilon_2 A, -)$. This homomorphism maps each n-fold exact sequence, representing an element of $Ext^t_A(\hat{S}, \hat{S})$, to the corresponding image representing an element of $Ext^t_{C_2}(\hat{S}\varepsilon_2, \hat{S}\varepsilon_2)$. In our situation, the homomorphism Ψ is surjective. Indeed, since the algebra $(A, \mathbf{e})$ is lean, $\mathrm{Ext}^1_A(\hat{S}\varepsilon_2 A, \hat{S}\varepsilon_2 A) \cong \mathrm{Ext}^1_{C_2}(\hat{S}\varepsilon_2, \hat{S}\varepsilon_2)$. Moreover, since $(A, \mathbf{e})$ is recursively Koszul, $\mathrm{Ext}^t_{C_2}(\hat{S}\varepsilon_2, \hat{S}\varepsilon_2) = (\mathrm{Ext}^1_{C_2}(\hat{S}\varepsilon_2, \hat{S}\varepsilon_2))^t$. Thus, Ψ is a surjection.

Now, the kernel $\mathrm{Ker}\,\Psi$ clearly contains $A^* f_1 A^*$. On the other hand, applying Lemma 2.2 (ii) to the module $X = \hat{S}$, and observing that $C_2{}^* = \mathrm{Ext}^*_{C_2}(X\varepsilon_2)$, we get that $\dim_K(A^*/A^* f_1 A^*) = \dim_K A^* - \dim_K A^* f_1 A^* = \dim_K C_2^*$. This shows that $\mathrm{Ker}\,\Psi = A^* f_1 A^*$, as required.

Finally, by induction, $(C_2{}^*, \mathbf{f}_2)$ is quasi-hereditary because C_2 is a recursively Koszul, lean and neat algebra. Thus $(A^*, \mathbf{f})$ is quasi-hereditary, as required. □

Let us point out that the above proof yields the following proposition.

PROPOSITION 2.4. *Let $(A, \mathbf{e})$ be a recursively Koszul and lean quasi-hereditary algebra. Then, for every $1 \leq i \leq n$, the extension algebra of the centralizer algebra C_i is isomorphic to the factor algebra $A^*/A^*\varphi_{i-1}A^*$, i. e. to the corresponding factor algebra of the quasi-hereditary algebra $(A^*, \mathbf{f})$.* □

Note that the dual statement holds for every quasi-hereditary algebra.

PROPOSITION 2.5. *Let $(A, \mathbf{e})$ be a quasi-hereditary algebra. Then the extension algebra of the factor algebra B_i is isomorphic to the centralizer algebra $\varphi_i A^* \varphi_i$ of the extension algebra A^* for every $1 \leq i \leq n$.*

Proof. If $(A, \mathbf{e})$ is quasi-hereditary then $\mathrm{Ext}^t_A(X, Y) \simeq \mathrm{Ext}^t_{B_i}(X, Y)$ for all $X, Y \in$ mod-B_i (cf. [CPS1] or [DR]), giving the required isomorphism. □

The following statement is an immediate consequence of Theorem 2.1.

THEOREM 2.6. *The extension algebra of a standard Koszul quasi-hereditary algebra is always quasi-hereditary (with respect to the opposite order).*

For standard Koszul quasi-hereditary algebras the functor Ext^* : mod-$A \to$ A^*-mod establishes a close connection between the standard modules of A and those of A^{*opp}.

PROPOSITION 2.7. *Let $(A, \mathbf{e})$ be a standard Koszul quasi-hereditary algebra. Then $\mathrm{Ext}^*_A(\Delta_A(i)) \cong \Delta^\circ_{A^*}(i)$, i. e. the right standard modules of $(A, \mathbf{e})$ are mapped to the left standard modules of $(A^*, \mathbf{f})$. Furthermore, the same correspondence maps the first syzygy and the radical of the standard module $\Delta_A(i)$ to the radical and the first syzygy of the left standard module $\Delta^\circ_{A^*}(i)$, respectively.*

Proof. Let us consider the exact sequences $0 \to V(i) \to P(i) \to \Delta(i) \to 0$ and $0 \to U(i) \to \Delta(i) \to S(i) \to 0$ of A-modules. We can apply Lemma 2.3 (ii) to both of these sequences to obtain the following exact sequences:

$$\begin{aligned} &0 \to \mathrm{Ext}^*_A(V(i)) \to \mathrm{Ext}^*_A(\Delta(i)) \to \mathrm{Ext}^*_A(P(i)) \to 0 \quad \text{and} \\ &0 \to \mathrm{Ext}^*_A(U(i)) \to \mathrm{Ext}^*_A(S(i)) \to \mathrm{Ext}^*_A(\Delta(i)) \to 0. \end{aligned} \tag{3}$$

Since the modules $V(i)$, $P(i)$ and $\Delta(i)$ all belong to $\mathcal{C}_A$, and $\mathrm{Ext}^*_A(P(i)) = S^\circ_{A^*}(i)$ obviously holds, $\mathrm{Ext}^*_A(\Delta(i))$ is a local module with top factor isomorphic to $S^\circ_{A^*}(i)$. Furthermore, since $\mathrm{Ext}^t_A(\Delta(i), S(j)) = 0$ for all $j < i$, the module $\mathrm{Ext}^*_A(\Delta(i))$ is an epimorphic image of $\Delta^\circ_{A^*}(i)$. In fact, for $i = 1$ we have $\mathrm{Ext}^*_A(\Delta(1)) = \mathrm{Ext}^*_A(S(1)) = P^\circ_{A^*}(1) = \Delta^\circ_{A^*}(1)$. Now, to prove the isomorphism for $i > 1$, it is enough to show that the K-dimensions of $\mathrm{Ext}^*_A(\Delta(i))$ and $\Delta^\circ_{A^*}(i)$ are equal. We shall proceed by induction on the number of simple modules. Since $\Delta_A(i) \in \mathcal{K}$, Lemma 2.2 (ii) yields $\dim_K \mathrm{Ext}^*_A(\Delta(i)) - \dim_K A^* f_1 \mathrm{Ext}^*_A(\Delta(i)) = \dim_K \mathrm{Ext}^*_{C_2}(\Delta(i)\varepsilon_2)$. Furthermore, by induction, $\mathrm{Ext}^*_{C_2}(\Delta(i)\varepsilon_2) = \mathrm{Ext}^*_{C_2}(\Delta_{C_2}(i)) \cong \Delta^\circ_{C_2^*}(i) \cong \Delta^\circ_{A^*/A^*f_1A^*}(i)$. Note that $A^* f_1 \mathrm{Ext}^*_A(\Delta(i)) = 0$ for $i > 1$, and thus $\dim_K \mathrm{Ext}^*_A(\Delta(i)) = \dim_K \Delta^\circ_{A^*/A^*f_1A^*}(i) = \dim_K \Delta^\circ_{A^*}(i)$, as required.

Finally, the exact sequences (3) show that $\mathrm{Ext}^*_A(V(i))$ is the radical of $\Delta^\circ_{A^*}(i)$, while $\mathrm{Ext}^*_A(U(i))$ is the first syzygy of $\Delta^\circ_{A^*}(i)$. □

While the extension algebra $(A^*, \mathbf{f})$ of a standard Koszul quasi-hereditary algebra $(A, \mathbf{e})$ is always quasi-hereditary (see Theorem 2.6), $(A^*, \mathbf{f})$ is not necessarily standard Koszul. In fact, it may even fail to be lean as the following proposition shows. Recall that we have already shown that the factor algebras $(B_i, \mathbf{e}_i)$ of a standard Koszul algebra are not necessarily standard Koszul (cf. Example 1.12).

PROPOSITION 2.8. *Let $(A, \mathbf{e})$ be a Koszul quasi-hereditary algebra. Then $(A^*, \mathbf{f})$ is lean if and only if all factor algebras B_i are Koszul.*

Proof. Observe first that $(A^*, \mathbf{f})$ is lean if and only if the species of the centralizer $\varphi_i A^* \varphi_i$ is the restriction of the species of A^* for all i. On the other hand the species of the factor algebra B_i is clearly the restriction of the species of A. Since A is Koszul, the species of A^* is the dual of the species of A. Thus $(A^*, \mathbf{f})$ is lean if and only if the species of the centralizer $\varphi_i A^* \varphi_i$ is the dual of the species of B_i for every i. In view of Proposition 2.5 this is equivalent to the condition that B_i is Koszul for every i. □

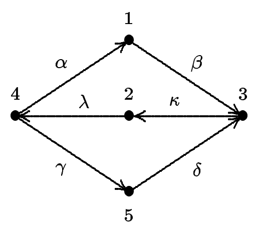
EXAMPLE 2.9. To illustrate the previous statement , let us consider once more the standard Koszul quasi-hereditary algebra A from Example 1.12. It was shown that the quotient algebra B_4 is not Koszul. The left regular representation of the extension algebra of A is given by

$$_{A^*}A^* = \begin{smallmatrix} & 1 & \\ 2 & & 5 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4 \oplus \begin{smallmatrix} 5 \\ 4 \end{smallmatrix},$$

which is not lean (with respect to the opposite order).

Let us conclude this section with yet another example showing that the correspondence between right standard modules of $(A, \mathbf{e})$ and left standard modules of $(A^*, \mathbf{f})$, given in Proposition 2.7 does not hold without the assumption that $(A, \mathbf{e})$ is quasi-hereditary.

EXAMPLE 2.10. Consider the algebra $A = KQ/I$, where Q is the quiver



and $I = \langle \lambda\alpha, \lambda\gamma, \kappa\lambda, \alpha\beta - \gamma\delta \rangle$; thus, the right and left regular representations are

$$A_A = \begin{smallmatrix} 1 \\ 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 4 & \\ 1 & & 5 \\ & 3 & \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 3 \\ 2 \end{smallmatrix} \quad \text{and} \quad {}_AA = \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ & 3 & \\ 1 & & 5 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 1 & & 5 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \end{smallmatrix}.$$

The algebra $(A, \mathbf{e})$ is a standard and recursively Koszul, lean and neat algebra, but it is not quasi-hereditary. The left regular representation of the extension algebra A^* is

$$_{A^*}A^* = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ & 4 & \\ 1 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 2 & \\ & 4 & \\ 1 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} & 4 & \\ 1 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}.$$

Hence, $(A^*, \mathbf{f})$ is a recursively Koszul quasi-hereditary algebra. It is neither standard Koszul, nor lean. In fact, the standard module $\Delta^\circ_{A^*}(2)$ is a proper (3-dimensional) homomorphic image of the 5-dimensional module $\mathrm{Ext}^*_A(\Delta_A(2)) = \mathrm{Ext}^*_A(S(2))$.

3. Quasi-hereditary extension algebras of graded algebras

In this final section we investigate the homological duality of finite dimensional algebras in the natural setting of (tightly) graded algebras, in which case the results of the previous section can be considerably strengthened. In particular, the extension algebra of a graded standard Koszul quasi-hereditary algebra is again standard Koszul and quasi-hereditary. Furthermore, if the extension algebra of a graded Koszul algebra is quasi-hereditary, then the original algebra is standard Koszul.

Here, by a *graded K-algebra A*, we shall understand a basic finite dimensional positively tightly graded algebra, i.e. $A = \oplus_{i\geq 0} A_i$ as a vector space, where A_0 is a semisimple K-algebra and $A_i \cdot A_j = A_{i+j}$ for all $i, j \geq 0$ (in particular, $A_i = (A_1)^i$ for all $i \geq 1$). Obviously, $\mathrm{rad}\, A = \oplus_{i\geq 1} A_i$. Let us fix a primitive orthogonal decomposition of the identity element $1 = e_1 + e_2 + \ldots + e_n$ so that $e_i \in A_0$ and define, for convenience, $A_j = 0$ for $j < 0$.

A finite dimensional *graded (right) A-module X* is a finite dimensional vector space $X = \oplus_{i\in\mathbb{Z}} X_i$ so that $X_i \cdot A_j \subseteq X_{i+j}$. A graded A-module X is said to be *generated in degree k*, if $X = X_k A$ (i. e. $X_k A_j = X_{k+j}$ for every integer j). Note that in this case $X_i = 0$ for all $i < k$. If $X = \oplus_{i\geq k} X_i$ is generated in degree k, then the graded submodule $\mathrm{rad}^j X = \oplus_{i\geq j+k} X_i$ is generated in degree $j+k$ for all $j > 0$.

In particular, the regular representation A_A and its indecomposable direct summands $P(i)$ are graded modules generated in degree 0 and the submodules $\mathrm{rad}^j A_A$ and $\mathrm{rad}^j P(i)$ in degree j for $j \geq 1$. Similarly, the simple top $S(i)$ of the indecomposable projective module $P(i)$ is generated in degree 0 as is the standard module $\Delta(i)$. Note that in the category of graded modules one also needs the *shifted projective modules* $P(i)[k]$ generated in degree k.

By a *(graded) morphism* $f : X \to Y$ between graded A-modules we shall understand a module homomorphism of degree 0, given by a family of linear maps $f_i : X_i \to Y_i$ with the obvious commuting properties. It is easy to see that for a graded morphism $f : X \to Y$, the morphisms $\mathrm{Ker}\, f \to X$ and $Y \to \mathrm{Coker}\, f$ are also graded, and if Y is generated in degree k then so is $\mathrm{Coker}\, f$. Every graded module X generated in degree k has a graded projective cover $P \to X$, where P is also generated in degree k.

Let X be a graded submodule of a graded module Y, and assume that Y is generated in degree k. Then $X \overset{t}{\subseteq} Y$ if and only if X is also generated in degree k. An immediate consequence is that for a graded a module X, generated in degree k, the module X is in $\mathcal{C}_A$ if and only if the j-th syzygy $\Omega_j(X)$ of X is generated in degree $k+j$ for every $j \geq 0$. This is clearly equivalent to the condition that in a minimal projective resolution $\cdots \to P_j \to \cdots \to P_1 \to P_0 \to X \to 0$ of X the projective module P_j is generated in degree $k+j$ for every $j \geq 0$, i. e. X has a *linear (graded) projective resolution.*

We start with the fact that, for a graded Koszul algebra A, the functor Ext^*_A maps a graded A-module generated in degree k which is in $\mathcal{C}_A$ to a graded A^*-module generated in degree k which is in $\mathcal{C}^\circ_{A^*}$ (cf. Proposition 5.1 of [GM2]). For a proof of this statement, we shall use the following lemma.

LEMMA 3.1. *Let A be a graded algebra and $0 \to X \to Y \to Z \to 0$ a short exact sequence of graded modules and graded morphisms such that X and Y (hence also Z) are generated in degree k. If X and Y belong to $\mathcal{C}_A$, then so does Z.*

Proof. Since both X and Y are generated in the same degree, hence $X \overset{t}{\subseteq} Y$, we have the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Omega(X) & \to & \Omega(Y) & \to & \Omega(Z) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & P(X) & \to & P(Y) & \to & P(Z) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X & \to & Y & \to & Z & \to & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Here $P(X), P(Y)$ and $P(Z)$ denote the projective covers of X, Y and Z, respectively. Since, by assumption, $X, Y \in \mathcal{C}_A$, the modules $\Omega(X)$ and $\Omega(Y)$ are generated in degree $k+1$. Hence, $\Omega(Z)$ is either 0 or generated in degree $k+1$. By induction on j, it follows that the j-th syzygy $\Omega_j(Z)$ of Z is generated in degree $k+j$ for every $j \geq 0$. Thus $Z \in \mathcal{C}_A$. □

COROLLARY 3.2. *Let A be a graded Koszul algebra and $X \in \mathcal{C}_A$ a graded module generated in degree k. Then $\mathrm{Ext}^*_A(X) \in \mathcal{C}^\circ_{A^*}$.*

Proof. For a given $X \in \mathcal{C}_A$ generated in degree k, let us consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & \Omega & \to & \mathrm{rad}\, P & \to & \mathrm{rad}\, X & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \Omega & \to & P & \to & X & \to & 0. \end{array}$$

Here, P is the projective cover of X and thus, since A is a Koszul algebra, $\mathrm{rad}\, P \in C_A$. Furthermore, both Ω and $\mathrm{rad}\, P$ are generated in degree $k+1$. Hence, Lemma 3.1 implies that $\mathrm{rad}\, X \in C_A$. By induction, $\mathrm{rad}^j X \in \mathcal{C}_A$ for every $j \geq 1$. Thus, by Proposition 3.5 of [ADL2], $\mathrm{Ext}^*(X) \in \mathcal{C}^\circ_{A^*}$, as required. □

THEOREM 3.3. *Let $(A, \mathbf{e})$ be a graded standard Koszul quasi-hereditary algebra. Then the extension algebra $(A^*, \mathbf{f})$ of A is also a standard Koszul quasi-hereditary algebra.*

Proof. In view of Theorem 2.6, we need to show only that the left and right standard modules of $(A^*, \mathbf{f})$ are in $\mathcal{C}^\circ_{A^*}$ and $\mathcal{C}_{A^*}$, respectively. Since, by Proposition 2.7, $\mathrm{Ext}^*_A(\Delta_A(i)) \cong \Delta^\circ_{A^*}(i)$ (and similarly for the left standard modules), the statement follows from Corollary 3.2. □

Note that another consequence of Corollary 3.2 is the well-known fact that the extension algebra of a graded Koszul algebra is a Koszul algebra (cf. e.g. [BGS]).

Let us observe that a crucial step in the proof of Corollary 3.2 is the fact that, for graded Koszul algebras, $X \in \mathcal{C}_A$ implies $\operatorname{rad} X \in \mathcal{C}_A$. Example 1.12 illustrates the fact that this implication does not hold in general for non-graded algebras. Clearly, $X = P(1)/P(1)\varepsilon_5 A$ has a top projective resolution, while $\operatorname{rad} X$ does not have one.

Theorems 2.1 and 2.6 are further strengthened in the following statement.

THEOREM 3.4. *Let $(A, \mathbf{e})$ be a graded Koszul algebra.*
(1) *$(A, \mathbf{e})$ is a recursively Koszul, neat and lean algebra if and only if $(A^*, \mathbf{f})$ is a quasi-hereditary algebra.*
(2) *$(A, \mathbf{e})$ is a quasi-hereditary algebra if and only if $(A^*, \mathbf{f})$ is a recursively Koszul, neat and lean algebra.*

Proof. Theorem 2.1 immediately implies one direction of (1). Furthermore, since for a Koszul algebra $A^{**} \cong A$ (cf. [BGS]), Theorem 2.1 also yields the opposite direction of (2). Hence, by the isomorphism above, it is enough to show that if $(A, \mathbf{e})$ is quasi-hereditary then $(A^*, \mathbf{f})$ is recursively Koszul, neat and lean.

To prove this, we shall need a few preparatory lemmas.

LEMMA 3.5. *Let $(A, \mathbf{e})$ be a quasi-hereditary graded algebra and X a graded A-module generated in degree k. Suppose that Xe_nA is a projective module. Then $X \in \mathcal{C}_A$ implies that $X/Xe_nA \in \mathcal{C}_{A/Ae_nA}$.*

Proof. We may asume that $k = 0$. Consider the linear projective resolution of X:

$$\cdots \to P_1 \to P_0 \to X \to 0.$$

Let us denote by Ω_i the i-th syzygy of X; let $\Omega_0 = X$.

First we are going to show by induction on i that the trace $\Omega_i e_n A$ of the projective module e_nA in the i-th syzygy is projective for every $i \geq 0$. The statement holds for $i = 0$ by assumption. The exact sequence $0 \to \Omega_{i+1} \to P_i \to \Omega_i \to 0$ yields the sequence

$$0 \to \Omega_{i+1} \cap P_i e_n A \to P_i e_n A \to \Omega_i e_n A \to 0.$$

Here $\Omega_i e_n A$ is projective by the induction hypothesis, and $P_i e_n A$ is projective by the quasi-heredity of A. So $\Omega_{i+1} \cap P_i e_n A$ is a direct summand of $P_i e_n A$. Thus $\Omega_{i+1} \cap P_i e_n A = (\Omega_{i+1} \cap P_i e_n A)e_n A = \Omega_{i+1} e_n A$ is a projective module.

Now we can consider the following diagrams with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega_{i+1} e_n A & \to & P_i e_n A & \to & \Omega_i e_n A & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega_{i+1} & \to & P_i & \to & \Omega_i & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \overline{\Omega}_{i+1} & \to & \overline{P}_i & \to & \overline{\Omega}_i & \to & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where all modules and morphisms are graded. Since P_i is generated in degree i, the module $\overline{P}_i$ is also generated in degree i. Clearly the grading of $\overline{X}$ and $\overline{P}_i$ as A-modules gives a grading over the graded algebra $B_{n-1} = A/Ae_nA$, so we obtain a linear projetive resolution of $\overline{X} \in$ mod-B_{n-1}. Thus $\overline{X} \in \mathcal{C}_{B_{n-1}}$, as required. □

This yields immediately the following corollary.

COROLLARY 3.6. *If $(A, \mathbf{e})$ is a graded quasi-hereditary Koszul algebra, then the factor algebras $B_i = A/A\varepsilon_{i+1}A$ are also Koszul algebras.*

Now, we return to the proof of Theorem 3.4. The quasi-heredity of $(A, \mathbf{e})$ implies that the centralizer algebras of $(A^*, \mathbf{f})$ are isomorphic to the homological dual of the factor algebras B_i which are Koszul by Corollary 3.6. As we observed earlier, the homological dual of a Koszul algebra is also Koszul. Thus the centralizer algebras of $(A^*, \mathbf{f})$ are Koszul, i. e. $(A^*, \mathbf{f})$ is recursively Koszul. Furthermore, Proposition 2.8 gives that $(A^*, \mathbf{f})$ is lean.

Since $(A, \mathbf{e})$ is quasi-hereditary, e_nA is Schurian in $A \cong A^{**}$. This means that $\mathrm{Ext}^i_{A^*}(S^{*\circ}(n), S^{*\circ}(n)) = 0$, for $i > 0$, showing that $f_n \in A^*$ is a neat idempotent. The rest now follows by induction on the number of simple modules. □

Let us formulate explicitly a few easy consequences of Theorem 3.4.

COROLLARY 3.7. *Let A be a graded Koszul algebra. Then $(A, \mathbf{e})$ is a standard Koszul quasi-hereditary algebra if and only if $(A^*, \mathbf{f})$ is a standard Koszul quasi-hereditary algebra.*

Note that Corollary 3.7 provides also an alternative proof of Theorem 3.3.

COROLLARY 3.8. *Let $(A, \mathbf{e})$ be a graded Koszul quasi-hereditary algebra. Then $(A, \mathbf{e})$ is a standard Koszul algebra if and only if $(A^*, \mathbf{f})$ is a quasi-hereditary algebra. Consequently, all algebras whose module categories are equivalent to the blocks of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ are standard Koszul.*

The second statement of Corollary 3.8 follows immediately from the result of Soergel [S] on self-duality of the algebras corresponding to regular blocks and from the result of Beilinson, Ginsburg and Soergel [BGS] that the module categories over the extension algebras of the algebras corresponding to singular blocks are categories studied by Rocha-Caridi in [R]. There it is shown that the respective algebras are quasi-hereditary. The authors are indebted to V. Mazorchuk for pointing out the latter reference to them.

A similar statement is valid for graded quasi-hereditary algebras with a Kazhdan–Lusztig theory in the sense of [CPS2].

Finally, let us formulate the counterpart of Proposition 1.11 for the factor algebras B_i. In view of Propositions 2.4 and 2.5, we have the following statement.

PROPOSITION 3.9. *Let $(A, \mathbf{e})$ be a a graded standard Koszul quasi-hereditary algebra. Then, for all $1 \le i \le n$, the factor and centralizer algebras $(B_i, \mathbf{e}_i)$ and $(C_i, \mathbf{e}_i)$ are also standard Koszul quasi-hereditary algebras; moreover,*

$$B_i{}^* \cong \varphi_i A^* \varphi \qquad \textit{and} \qquad C_i{}^* \cong A^*/A^*\varphi_{i-1}A^*.$$

Proof. Most of the statements are contained in Propositions 1.11, 2.4, 2.5 and the proof of Theorem 3.4. The only part left to prove is that the algebras $(B_i, \mathbf{e}_i)$ are standard Koszul. By Corollary 3.6 the algebras B_i are Koszul, and since forming the centralizer algebras and factor algebras commutes, the same corollary yields that the algebras $(B_i, \mathbf{e}_i)$ are recursively Koszul. Finally, Theorem 1.4 implies that $(B_i, \mathbf{e}_i)$ are standard Koszul for all $1 \leq i \leq n$. □

References

[ADL1] Ágoston, I., Dlab, V., Lukács, E., Lean algebras, in: *Representations of algebras. Sixth International Conference, 1992, Ottawa.* CMS Conference Proceedings **14** (1993), 1–14.

[ADL2] Ágoston, I., Dlab, V., Lukács, E., Homological duality and quasi-heredity, *Can. J. Math.* **48** (1996), 897–917.

[ADL3] Ágoston, I., Dlab, V., Lukács, E., Stratified algebras, *C.R.Math.Rep. Acad.Sci.Canada* **20** (1998), 20–25.

[ADW] Ágoston, I., Dlab, V., Wakamatsu, T., Neat algebras, *Comm. Alg.* **19** (1991), 433–442.

[BG] Beilinson, A., Ginzburg, V., Mixed categories, Ext-duality and representations (results and conjectures), preprint, Moscow, 1986.

[BGS] Beilinson, A., Ginzburg, V., Soergel, W., Koszul duality patterns in representation theory, *J. Amer. Math. Soc.*, **9** (1996), 473–527.

[BGG] Berstein, I.N., Gelfand, I.M., Gelfand, S.I., A category of $\mathfrak{g}$-modules, *Funct. Anal. Appl.* **10** (1976), 87–92.

[CPS1] Cline, E., Parshall, B.J., Scott, L.L., Finite dimensional algebras and highest weight categories, *J. Reine Angew. Math.* **391** (1988) 85–99.

[CPS2] Cline, E., Parshall, B.J., Scott, L.L., The homological dual of a highest weight category, *Proc. London Math. Soc.* **68** (1994), 294–316.

[CPS3] Cline, E., Parshall, B., Scott, L.L., *Stratifying Endomorphism Algebras.* Memoirs of the AMS **591** (1996)

[CPS4] Cline, E., Parshall, B., Scott, L.L., Graded and non-graded Kazhdan-Lusztig theories, in: *Algebraic groups and Lie groups.* Austral. Math. Soc. Lect. Ser. **9** (1997), 105–125.

[DR] Dlab, V., Ringel, C.M., Quasi-hereditary algebras, *Ill. Journal of Math.* **35** (1989), 280–291.

[GM1] Green, E.L., Martínez-Villa, R., Koszul and Yoneda algebras, in: *Representation theory of algebras. Seventh International Conference, 1994, Cocoyoc, Mexico.* CMS Conference Proceedings **18** (1995), 247–297.

[GM2] Green, E.L., Martínez-Villa, R., Koszul and Yoneda algebras II, in: *Algebras and modules. Eight International Conference on representations of algebras, 1996, Gerainger, Norway.* CMS Conference Proceedings **24** (1998), 227–244.

[P1] PARSHALL, B., Koszul algebras and duality, CMS Conference Proceedings **16** (1995), 277–285.

[P2] PARSHALL, B., Some finite dimensional algebras arising in group theory, in: *Algebras and modules. Workshop on representation theory of algebras and related topics, 1996, Trondheim, Norway.* CMS Conference proceedings **23** (1998), 107–156.

[PS1] PARSHALL, B.J., SCOTT, L.L., Derived categories, quasi-hereditary algebras and algebraic groups, *Proc. Ottawa-Moosonee Workshop in Algebra,.* Carleton-Ottawa Math Lecture Notes **3** (1988), 1–105.

[PS2] PARSHALL, B.J., SCOTT, L.L., Koszul algebras and the Frobenius automorphism, *Quart.J.Math Oxford (2)*, **46** (1995), 345–384.

[R] ROCHA-CARIDI, A., Splitting criteria for g-modules induced from a parabolic and the Bernstein-Gelfand-Gelfand resolution of a finite-dimensional, irreducible g-module, *Trans. Amer. Math. Soc.* **262** (1980), 335–366.

[S] SOERGEL, W., Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe, *J. Amer. Math. Soc.*, **3** (1990), 421–445.

DEPARTMENT OF ALGEBRA AND NUMBER THEORY, EÖTVÖS UNIVERSITY, H-1053 BUDAPEST, HUNGARY
E-mail: agoston@cs.elte.hu

SCHOOL OF MATHEMATICS AND STATISTICS, CARLETON UNIVERSITY, OTTAWA, ONTARIO, CANADA, K1S 5B6
E-mail: vdlab@math.carleton.ca

DEPARTMENT OF ALGEBRA, TECHNICAL UNIVERSITY OF BUDAPEST, P.O.BOX 91, H-1521 BUDAPEST, HUNGARY
E-mail: lukacs@math.bme.hu