P-BASES FOR TORSION-FREE REGULAR MODULES

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Abstract. In [**R**] a general approach is offered to the theory of infinite dimensional representations over tame hereditary algebras. Among other things many concepts, like that of torsion and torsion-free modules, divisibility, etc., known from the theory of abelian groups, are carried over to the tame hereditary situation. In this approach one of the most important invariants of torsion-free modules is their rank. In our short note we will show that the rank of the torsion-free regular module M, originally defined by Ringel using a certain embedding of M into a divisible module, can be understood as the cardinality of a maximal independent set for a suitably defined dependence relation.

1. Preliminaries. First we recall some definitions and basic results from [**R**]. Let A be a tame hereditary algebra, finite dimensional over a field k. One can define a torsion theory on the category of (right) A-modules as follows. A module M is called *torsion* if it is spanned by its finite dimensional regular and preinjective submodules, while M is *torsion-free* if every finite dimensional submodule of M is preprojective. The *torsion submodule* of a module M is the largest submodule which is torsion and it will be denoted by $\mathcal{T}(M)$. We call a module M *regular* if it has no finite dimensional preinjective or preprojective direct summands. It can be shown that a module M is regular if and only if $\operatorname{Hom}(M, P) = 0$ for all finite dimensional preprojective modules P and $\operatorname{Hom}(I, M) = 0$ for all finite dimensional preinjective modules I (cf. §4.2 of [**R**]). The class of torsion regular modules over A forms an exact abelian subcategory, closed under extensions (Theorem 4.4 of [**R**]).

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A module M is called *divisible* if $\operatorname{Ext}(S, M) = 0$ for every simple regular module S. Actually, if M is divisible then $\operatorname{Ext}(N, M) = 0$ for all regular modules N (Proposition 4.7 of [**R**]). As Ringel has shown, there exists a unique indecomposable torsion-free divisible module, denoted by Q (Theorem 5.3 of [**R**]). The endomorphism ring of Q is a division ring. Every torsion-free module M can be embedded into a direct sum Y of copies of Q in such a way that the quotient Y/M is torsion regular. Moreover, if there is such an embedding into $Y = \bigoplus_I Q$ then the cardinality of the index set I depends only on the module M (Theorem 5.5 of [**R**]). This cardinality is called the *rank* of M and will be denoted by $\operatorname{rk} M$. One can show that the rank is additive on short exact sequences of torsion-free modules (Proposition 2.2 of [**DZ**]). An important fact is that for finite dimensional preprojective modules the rank is just the negative of the defect of the module (Proposition 5.6 of [**R**]).

A submodule N of the module M is called *torsion closed* in M if M/N is torsion-free. The intersection of all torsion-closed submodules of M containing a particular submodule N is called the *torsion-closure* of N in M and will be denoted by $\overline{N}^M$ or simply by $\overline{N}$. Clearly $\overline{N}^M$ is the full preimage of $\mathcal{T}(M/N)$ in M. For any submodule N of M the rank of $\overline{N}$ must satisfy $\operatorname{rk} \overline{N} \leq \operatorname{rk} N$ (Proposition 2.1 of [**ADS**]). The following simple observation will show that taking the torsion closure of a module is of finitary character.

Lemma 1.1. *Let $\mathcal{N} = \{N_i \mid i \in I\}$ be a directed set of torsion-closed submodules of a module M. Then $N = \bigcup_{i \in I} N_i \leq M$ is also torsion-closed in M.*

Proof. Let us take an arbitrary submodule $W \leq M$ such that $N \leq W$ and W/N is finite dimensional. We have to show that W/N is preprojective. Let W' be a finite dimensional preimage of W/N in M, i.e. $W' \leq M$ such that $W' + N = W$. Consider $W' \cap N = W' \cap (\bigcup_{i \in I} N_i) = \bigcup_{i \in I} (N_i \cap W')$. Since W' is finite dimensional and $\mathcal{N}$ is directed, there exists an index $i \in I$ such that $W' \cap N = W' \cap N_i$. Then:

$$W/N = (W' + N)/N \cong W'/W' \cap N = W'/W' \cap N_i \cong (W' + N_i)/N_i \leq M/N_i.$$

Since M/N_i is torsion-free by assumption, W/N is preprojective. Hence N is torsion-closed.

Corollary 1.2. *If $\mathcal{N} = \{N_i \mid i \in I\}$ is a directed set of submodules of a module M then $\overline{\bigcup_{i \in I} N_i} = \bigcup_{i \in I} \overline{N_i}$. In particular for any submodule N of M the* *torsion closure* $\overline{N}$ *is the union of the torsion closures of all finite dimensional submodules of* N.

2. A dependence relation for torsion-free regular modules. For the rest of the paper, unless otherwise stated, let M be a torsion-free regular module. We shall define a dependence relation on a subset of all submodules of M. Let P be an indecomposable projective module of rank 1 (that is, of defect -1). Define the set $\mathcal{W}_P = \mathcal{W}_P(M)$ as $\mathcal{W}_P = \{W \leq M \mid W \cong P\}$. For $W \in \mathcal{W}_P$ and $\mathcal{P} \subseteq \mathcal{W}_P$ we say that W *depends* on $\mathcal{P}$ if and only if $W \subseteq \overline{\mathcal{P}}^M$, where $\overline{\mathcal{P}}^M$ denotes the torsion closure of the submodule generated by the elements of $\mathcal{P}$. We call a set $\mathcal{P} \subseteq \mathcal{W}_P$ *independent* if no element $W \in \mathcal{P}$ depends on $\mathcal{P} \setminus \{W\}$. The set $\mathcal{P} \subseteq \mathcal{W}_P$ is called a *generating set* for $\mathcal{W}_P$ if every element of $\mathcal{W}_P$ depends on $\mathcal{P}$.

We want to show that this relation satisfies all the standard properties of linear dependence relations. We will need the following observation.

Lemma 2.1. *Let* $W \in \mathcal{W}_P$ *and* $\mathcal{P} \subseteq \mathcal{W}_P$. *Then* W *depends on* $\mathcal{P}$ *if and only if* $W \cap \overline{\mathcal{P}} \neq 0$.

Proof. The necessity of the condition is obvious from the definition. To show the other direction, assume that $W \cap \overline{\mathcal{P}} \neq 0$. Observe that $\operatorname{rk} W = 1$ implies $\operatorname{rk} \overline{W} = 1$. Since $\overline{W} \cap \overline{\mathcal{P}}$ is torsion closed in $\overline{W}$ and is non-zero by assumption, we must have that $\overline{W} \subseteq \overline{\mathcal{P}}$. Hence W depends on $\mathcal{P}$.

The next proposition shows that our dependence relation is of finitary character.

Proposition 2.2. *Let* W *be an element of* $\mathcal{W}_P$. *If* W *depends on* $\mathcal{P} \subseteq \mathcal{W}_P$ *then it also depends on a finite subset* $\mathcal{F} \subseteq \mathcal{P}$.

Proof. Assume that $W \subseteq \overline{\mathcal{P}}$. Since $\dim_k W < \infty$, by Corollary 1.2 we get that there is a finite dimensional submodule $N \subseteq \langle \mathcal{P} \rangle$ such that $W \subseteq \overline{N}$. Since there is a finite subset $\mathcal{F} \subseteq \mathcal{P}$ such that $N \subseteq \langle \mathcal{F} \rangle$, we obtain that W depends on $\mathcal{F}$.

Finally, we can show that the exchange property is also satisfied.

Proposition 2.3. *Let* $W \in \mathcal{W}_P$ *and* $\mathcal{P} \subseteq \mathcal{W}_P$. *If* W *depends on* $\mathcal{P}$ *but does not depend on* $\mathcal{P}' = \mathcal{P} \setminus \{X\}$ *for some element* $X \in \mathcal{P}$ *then* X *depends on* $\mathcal{P}'' = (\mathcal{P} \setminus \{X\}) \cup \{W\}$.

Proof. By assumption we have $\overline{\mathcal{P}'} \subset \overline{\mathcal{P}''} \subseteq \overline{\mathcal{P}}$, hence X does not depend on $\mathcal{P}'$. Thus Lemma 2.1 implies that X embeds into $\overline{\mathcal{P}}/\overline{\mathcal{P}'}$ and the torsion closure of its image is the whole module $\overline{\mathcal{P}}/\overline{\mathcal{P}'}$. Hence we get that $\operatorname{rk} \overline{\mathcal{P}}/\overline{\mathcal{P}'} = 1$. Consequently, we must have $\overline{\mathcal{P}''} = \overline{\mathcal{P}}$. Thus $X \subseteq \overline{\mathcal{P}''}$, that is, X depends on $\mathcal{P}''$.

From the exchange property and the finitary character of our dependence relation we get all the standard theorems on the existence of bases, the invariance of their cardinality etc. A basis for $\mathcal{W}_P$ with respect to this relation will be called a *P-basis* for M.

3. Independent sets, generating sets and the rank. We turn now to the question of characterizing the specific properties of our dependence relation.

Theorem 3.1. *A set $\mathcal{P} = \{P_i \mid i \in I\} \subseteq \mathcal{W}_P$ is independent if and only if*
(i) $P_i \cap \langle P_j \in \mathcal{P} \mid j \neq i \rangle = 0$ for every $i \in I$ (that is, $\bigoplus_{i \in I} P_i \subseteq M$);
(ii) $M/\bigoplus_{i \in I} P_i$ is regular.

Proof. Assume first that the conditions (i) and (ii) are satisfied, thus we have $M/\bigoplus_{i \in I} P_i$ regular. Then consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{i \in I} P_i & \xrightarrow{\iota} & M & \longrightarrow & M/\bigoplus_{i \in I} P_i & \longrightarrow & 0 \\ & & \downarrow \pi_i & \swarrow \psi & & & & & \\ & & P_i & & & & & & \\ & & \downarrow \iota_i & & & & & & \\ & & Q & & & & & & \end{array}$$

where π_i is the canonical projection and ι_i is an embedding of P_i into Q. Since $M/\bigoplus_{i \in I} P_i$ is regular, $\operatorname{Ext}(M/\bigoplus_{i \in I} P_i, Q) = 0$, hence there exists a map $\psi : M \to Q$ such that $\psi\iota = \iota_i \pi_i$. Thus $\operatorname{Ker}\psi \supseteq \overline{\mathcal{P} \setminus \{P_i\}}$ and $\operatorname{Ker}\psi \cap P_i = 0$. So $\mathcal{P}$ is independent.

Assume now that $\mathcal{P} \subseteq \mathcal{W}_P$ is independent. Thus by Lemma 2.1 we get that $P_i \cap \langle P_j \in \mathcal{P} \mid j \neq i \rangle \subseteq P_i \cap \overline{(\mathcal{P} \setminus \{P_i\})} = 0$. Hence we get (i). To prove condition (ii), assume $\mathcal{P} = \{P_\mu \mid \mu < \kappa\}$ for some cardinal κ. We will prove by transfinite induction that $M_\lambda = M/\bigoplus_{\mu < \lambda} P_\mu$ is regular for each $\lambda \leq \kappa$. The argument we use is essentially from Ringel's proof of Proposition 4.3 of [**R**], but we include it here for the sake of completeness.

Assume first that $\lambda = \nu + 1$ for some ordinal ν. By assumption $M_\nu = M/\bigoplus_{\mu<\nu} P_\mu$ is regular. Since $\mathcal{P}$ is independent, by Lemma 2.1 we get that P_ν embeds into M_ν as $\hat{P}_\nu$; actually it embeds even into the torsion-free module $\tilde{M}_\nu = M_\nu/\mathcal{T}(M_\nu)$ as $\tilde{P}_\nu$. We will show first that $\tilde{M}_\nu/\tilde{P}_\nu$ is regular. Since M was regular, it is enough to check that $\tilde{M}_\nu/\tilde{P}_\nu$ has no preinjective submodules. Let us assume that $\tilde{W} \subseteq \tilde{M}_\nu$ is a submodule containing $\tilde{P}_\nu$ and $\tilde{W}/\tilde{P}_\nu$ is indecomposable preinjective. But then for the defect of $\tilde{W}$ we get: $\delta(\tilde{W}) = \delta(\tilde{P}_\nu) + \delta(\tilde{W}/\tilde{P}_\nu) = -1 + \delta(\tilde{W}/\tilde{P}_\nu) \geq 0$, and this contradicts the fact that $\tilde{M}_\nu$ is torsion-free. Thus $\tilde{M}_\nu/\tilde{P}_\nu$ is regular. Consider now the following exact sequence:

$$0 \longrightarrow (\mathcal{T}(M_\nu) + \hat{P}_\nu)/\hat{P}_\nu \longrightarrow M_\nu/\hat{P}_\nu \longrightarrow (M_\nu/\hat{P}_\nu)/\big((\mathcal{T}(M_\nu) + \hat{P}_\nu)/\hat{P}_\nu\big) \longrightarrow 0.$$

Here the first term is isomorphic to $\mathcal{T}(M_\nu)$, and since it is the torsion submodule of the regular module M_ν, it is regular. On the other hand the last term is isomorphic to $\tilde{M}_\nu/\tilde{P}_\nu$, which was just shown to be regular, too. Thus $M_\nu/\hat{P}_\nu \cong M/\bigoplus_{\mu<\lambda} P_\mu = M_\lambda$, as an extension of two regular modules, is also regular.

Assume now that λ is a limit ordinal and for every $\mu < \lambda$ the module M_μ is regular. Suppose that $M_\lambda = M/\bigoplus_{\mu<\lambda} P_\mu$ has an indecomposable preinjective submodule $W/\bigoplus_{\mu<\lambda} P_\mu$ where $\bigoplus_{\mu<\lambda} P_\mu \subseteq W \subseteq M$. Let W' be a finite dimensional submodule of M satisfying $W = W' + \bigoplus_{\mu<\lambda} P_\mu$. Then $W' \cap \bigoplus_{\mu<\lambda} P_\mu = W' \cap \bigoplus_{\mu<\nu} P_\mu$ for some $\nu < \lambda$. Thus:

$$\begin{aligned} W/\bigoplus_{\mu<\lambda} P_\mu &= \big(W' + \bigoplus_{\mu<\lambda} P_\mu\big)/\bigoplus_{\mu<\lambda} P_\mu \cong W'/\big(W' \cap \bigoplus_{\mu<\lambda} P_\mu\big) = \\ &= W'/\big(W' \cap \bigoplus_{\mu<\nu} P_\mu\big) \cong \big(W' + \bigoplus_{\mu<\nu} P_\mu\big)/\bigoplus_{\mu<\nu} P_\mu \subseteq M_\nu, \end{aligned}$$

and this would contradict the regularity of M_ν. Thus M_λ is also regular. This finishes the proof.

Theorem 3.2. *A subset $\mathcal{P} \subseteq \mathcal{W}_P$ is a generating set if and only if $M/\langle\mathcal{P}\rangle$ is torsion, i.e. if and only if $\overline{\mathcal{P}} = M$.*

Proof. We will again use the argument of Ringel from the proof of Proposition 4.3 of [**R**]. The sufficiency is obvious from the definitions. For the other direction let us assume that $\mathcal{P}$ is a generating set. Assume that $\overline{\mathcal{P}} \neq M$. Then the module $\tilde{M} = M/\overline{\mathcal{P}}$ is a non-zero torsion-free regular module. But then there must exist a non-zero homomorphism $\varphi : P \to \tilde{M}$, otherwise $\tilde{M}$ would become a module over an algebra of finite representation type, and as such it would be a direct sum of finite dimensional indecomposable submodules, contradicting the fact that $\tilde{M}$ is torsion-free and regular. Since $\operatorname{rk} P = 1$, we get that φ must be an embedding. But P is projective, so this also gives an embedding of P into M and the image P_0 is disjoint from $\overline{\mathcal{P}}$. Thus P_0 does not depend on $\mathcal{P}$, contradicting the assumption that $\mathcal{P}$ was a generating set for M.

Finally we want to show that the cardinality of a P-basis for M is $\operatorname{rk} M$.

Theorem 3.3. *Let $\mathcal{P} = \{P_i \mid i \in I\} \subseteq \mathcal{W}_P$ be an independent set. Then $\operatorname{rk} \overline{\mathcal{P}} = |\mathcal{P}|$. In particular if $\mathcal{P}$ is a P-basis for M then $|\mathcal{P}| = \operatorname{rk} M$.*

Proof. Let us take $N = \overline{\mathcal{P}}$. Then the module $N/ \bigoplus_{i\in I} P_i$ is the torsion submodule of the module $M/ \bigoplus_{i\in I} P_i$, and the latter is regular by Theorem 3.1. Thus $N/ \bigoplus_{i\in I} P_i$ must also be regular. If we now embed the module N into a direct sum Y of copies of the module Q in such a way that Y/N is torsion regular, then $Y/ \bigoplus_{i\in I} P_i$ is also torsion regular. Hence $\operatorname{rk} N = \operatorname{rk} \bigoplus_{i\in I} P_i$. But clearly $\operatorname{rk} \bigoplus_{i\in I} P_i = |I| = |\mathcal{P}|$, thus we are done.

The previous result shows that the cardinality of a P-basis for M would be the same if we have started with another indecomposable projective module P' which is of rank 1. As a matter of fact we did not use that our set $\mathcal{W}_P$ was "homogeneous": it would have been possible to define our dependence relation for the set $\mathcal{W} = \mathcal{W}(M) = \{N \subseteq M \mid N \text{ is indecomposable projective}, \operatorname{rk} N = 1\}$.

4. Further results. Let us take an independent set $\mathcal{P} = \{P_i \mid i \in I\} \subseteq \mathcal{W}_P$. Then by Theorem 3.1 we get that $\bigoplus_{i\in I} P_i \subseteq M$ and $M/ \bigoplus_{i\in I} P_i$ is regular. If we take a projection map $\pi_i : \bigoplus_{i\in I} P_i \to P_i$ and compose it with an embedding $\iota_i : P_i \to Q$ then we can extend this map to a homomorphism $\psi_i : M \to Q$. Let us choose such a homomorphism ψ_i for each $i \in I$. Then we have the following proposition.

Proposition 4.1. *Let $\mathcal{P} = \{P_i \mid i \in I\} \subseteq \mathcal{W}_P$ be an independent set for the torsion-free module M and $\Psi = \{\psi_i \mid i \in I\} \subseteq \operatorname{Hom}(M, Q)$ be a set of homomorphisms as defined above. Then Ψ is independent over* $\operatorname{End}(Q)$.

Proof. Assume that $\alpha_{i_1}\psi_{i_1} + \cdots + \alpha_{i_n}\psi_{i_n} = 0$ for some elements $\alpha_{i_1}, \ldots, \alpha_{i_n} \in \operatorname{End}(Q)$ and some indices $i_1, \ldots, i_n \in I$. We may assume that the indices are all distinct. Since $P_{i_j} \subseteq \operatorname{Ker}\psi_{i_\ell}$ for $j \neq \ell$, we get that $P_{i_j} \subseteq \operatorname{Ker}\alpha_{i_j}\psi_{i_j}$ for $1 \leq j \leq n$. But since $\operatorname{Ker}\psi_{i_j} \cap P_{i_j} = 0$, we must have that $\operatorname{Ker}\alpha_{i_j} \neq 0$ so $\alpha_{i_j} = 0$ for every $1 \leq j \leq n$. Thus Ψ is independent over $\operatorname{End}(Q)$.

As an application of our concepts we will conclude with a statement which generalizes Proposition 6.1.2 of [**R**].

Proposition 4.2. *Let M be a torsion-free (not necessarily regular) module of finite rank, and let $N \subseteq M$ be a submodule such that M/N is torsion. Then M/N is regular if and only if* $\operatorname{rk} N = \operatorname{rk} M$.

Proof. Assume first that M/N is regular. Then an embedding of M into a direct sum $\underset{i\in I}{\oplus} Q$ with a torsion regular cokernel yields an embedding of N into the same direct sum with a cokernel term which is also torsion regular. Hence $\operatorname{rk} N = \operatorname{rk} M$.

Assume now that $\operatorname{rk} N = \operatorname{rk} M = n$. Then we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\iota_N} & \overset{n}{\underset{1}{\oplus}} Q = Y' & \longrightarrow & Y'/N & \longrightarrow & 0 \\ & & \big\downarrow \varphi' & & \big\downarrow \varphi & & \big\downarrow \varphi'' & & \\ 0 & \longrightarrow & M & \xrightarrow{\iota_M} & \overset{n}{\underset{1}{\oplus}} Q = Y'' & \longrightarrow & Y''/M & \longrightarrow & 0. \end{array}$$

Here φ' is the embedding of N into M, the modules Y'/N and Y''/M are torsion regular, and the existence of φ follows from $\operatorname{Ext}(Y'/N, Y'') = 0$. Since the modules M/N and Y''/M are torsion, we get that Y''/N is also torsion, hence $\overline{\operatorname{Im}\varphi}^{Y''} = Y''$. This immediately gives that φ is a monomorphism, as otherwise the exact sequence

$$0 \longrightarrow \operatorname{Ker}\varphi \longrightarrow Y' \longrightarrow \operatorname{Im}\varphi \longrightarrow 0$$

would give that $\operatorname{rk}\operatorname{Im}\varphi < \operatorname{rk} Y' = n$, and then we would have that $\operatorname{rk}\overline{\operatorname{Im}\varphi}^{Y''} \leq \operatorname{rk}\operatorname{Im}\varphi < n = \operatorname{rk} Y''$, which contradicts the fact that $\overline{\operatorname{Im}\varphi}^{Y''} = Y''$.

To prove that φ is an epimorphism we will show first that $\operatorname{Cok}\varphi$ is torsion regular. Take a P-basis $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ for Y'. Then it embeds into Y'' as $\varphi(\mathcal{P}) = \{\varphi(P_1), \varphi(P_2), \ldots, \varphi(P_n)\} \subseteq \mathcal{W}_P(Y'')$. Since $\overline{\operatorname{Im}\varphi}^{Y''} = Y''$, we must have $\overline{\varphi(\mathcal{P})}^{Y''} = Y''$, so Theorem 3.2 implies that $\varphi(\mathcal{P})$ is a generating set for $\mathcal{W}_P(Y'')$. Since $\operatorname{rk} Y'' = n = |\varphi(\mathcal{P})|$, we get that $\varphi(\mathcal{P})$ is a P-basis for Y''. So $Y''/\langle\varphi(\mathcal{P})\rangle$ is torsion regular by Theorem 3.1. Hence $\operatorname{Cok}\varphi \cong (Y''/\langle\varphi(\mathcal{P})\rangle)/\operatorname{Im}\varphi/\langle\varphi(\mathcal{P})\rangle$ is also torsion regular.

From the fact that $\operatorname{Cok}\varphi$ is torsion regular we get that $\operatorname{Ext}(\operatorname{Cok}\varphi, \operatorname{Im}\varphi) = 0$ hence the embedding $\varphi : Y' \to Y''$ splits. As Y'' is torsion-free, this implies that $\operatorname{Cok}\varphi = 0$, i.e. φ is an epimorphism.

Finally, since $\operatorname{Cok}\varphi = 0$, the Snake Lemma gives us that $\operatorname{Ker}\varphi'' \cong \operatorname{Cok}\varphi' = M/N$. Since φ'' is a homomorphism between torsion regular modules, $\operatorname{Ker}\varphi''$ is torsion regular. Hence M/N is torsion regular, as required.

It is easy to construct examples to show that the assumption on the finiteness of $\operatorname{rk} N = \operatorname{rk} M$ is necessary. Take for instance a module Q_1 isomorphic to Q with two disjoint non-zero finite dimensional submodules: $P_0, P_1 \subseteq Q_1$. (Since $\operatorname{Soc} Q_1$ is not simple, one can obviously find such submodules.) Let us also choose arbitrary non-zero finite dimensional submodules $P_i \subseteq Q_i \cong Q$ for $i = 2, 3, \ldots$. Take $M = \bigoplus_{i=1}^{\infty} Q_i$, $N = \bigoplus_{i=0}^{\infty} P_i$. Then obviously $\operatorname{rk} N = \operatorname{rk} M = \aleph_0$ with $\overline{N} = M$. But M/N cannot be regular as otherwise one could extend the homomorphism $\bigoplus_{i=0}^{\infty} P_i \xrightarrow{\pi_0} P_0 \xrightarrow{\iota} Q$ to a homomorphism $\bigoplus_{i=1}^{\infty} Q_i \xrightarrow{\psi} Q$. Then $\operatorname{Ker}\psi \supseteq \bigoplus_{i=1}^{\infty} P_i$, hence $\operatorname{Ker}\psi \supseteq \overline{\bigoplus_{i=1}^{\infty} P_i}^{M} = M$, contradicting the fact that $\psi \neq 0$, since for example $\operatorname{Ker}\psi \cap P_0 = 0$.

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