NEAT ALGEBRAS

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§0. Introduction

In a recent paper [**DR1**], Dlab and Ringel produced a ring construction by which they described explicitly the categories of perverse sheaves studied by Mirollo and Vilonen [**MV**], and presented an inductive construction of quasi-hereditary rings. Inspired by their approach, we consider in this paper finite dimensional (associative) K-algebras A which are *smoothly constructed* in the following sense: there is an order $(e_1, e_2, \dots, e_m)$ of a complete set of (primitive) orthogonal idempotents of A such that for every t, $1 \le t \le m-1$, the multiplication map

$$e_t A \varepsilon_{t+1} \underset{\varepsilon_{t+1} A \varepsilon_{t+1}}{\otimes} \varepsilon_{t+1} A e_t \to e_t A \varepsilon_{t+1} A e_t,$$

where $\varepsilon_t = e_t + e_{t+1} + \dots + e_m$ (and $\varepsilon_{m+1} = 0$), is bijective. Let us remark that the algebras obtained by iterating the $A(\gamma)$ construction of [**DR1**] in which D is always a semisimple algebra and the initial $C = 0$, as well as the quasi-hereditary algebras satisfy this condition. Of course, every quasi-hereditary algebra A satisfies a stronger condition: it possesses a **t**-*sequence* $(e_1, e_2, \dots, e_m)$, i.e. a complete sequence of (primitive) orthogonal idempotents such that, for every t, $1 \le t \le m-1$, the multiplication map

$$A \varepsilon_{t+1} \underset{\varepsilon_{t+1} A \varepsilon_{t+1}}{\otimes} \varepsilon_{t+1} A \to A \varepsilon_{t+1} A$$

is bijective [**DR1**]. An equivalent formulation of this property, related to [**DR2**] is given in Corollary 5.

The main objective of our study are the *neat algebras.* A finite dimensional algebra A is *neat* if it possesses a *neat sequence* $(e_1, e_2, \ldots, e_m)$, i.e. a complete sequence of (primitive) orthogonal idempotents such that, for every $t, 1 \leq t \leq m$,

$$Ext^i_{\varepsilon_t A \varepsilon_t}(S(e_t), S(e_t)) = 0$$

for all $i \geq 1$; here $S(e_t)$ denotes the semisimple (right) $\varepsilon_t A \varepsilon_t$-module $e_t A \varepsilon_t / e_t Rad A \varepsilon_t$. Equivalently, no term of the minimal projective $\varepsilon_t A \varepsilon_t$-resolution of $e_t Rad A \varepsilon_t$ has a summand isomorphic to a summand of $e_t A \varepsilon_t, 1 \leq t \leq m$ (see Proposition 1 in §1). Let us note here, that all of the above mentioned properties are well preserved under *refinements* of the sequence, and thus we do not have to assume in our definitions that the idempotents are primitive.

All quasi-hereditary algebras are neat; in fact, every heredity sequence of idempotents is neat [**DR2**]. The converse does not hold in general (see §2, Example 3). The main result of the paper is the following.

Theorem. *The global dimension of a neat algebra is finite.*

The proof of the theorem, including a bound on the global dimension of a neat algebra is given in §1, Corollary 4.

Every neat sequence is an **r**-*sequence* in the following sense. For every $t, 1 \leq t \leq m$,

$$e_t A \varepsilon_{t+1} A e_t = e_t Rad A e_t$$

(see §1, Corollary 1.). Since an **r**-sequence which is also a **t**-sequence is clearly a heredity sequence (cf. [**DR1**]), we conclude the following:

Corollary. *An algebra is quasi-hereditary if and only if it posseses a neat* **t**-*sequence.*

The second part of the paper contains some additional remarks concerning the best estimates for the global dimension in Proposition 2, the relationship between neat sequences, **t**-sequences and global dimension, an alternative proof of the fact, that every algebra is the endomorphism ring of a projective module over a quasi-hereditary algebra (cf. [**DR3**]), as well as the fact that the determinant of the Cartan matrix of a neat algebra is 1.

The results of this paper were reported by the first author in the Workshop on Representation Theory held in Ottawa, on 20–22 April, 1989.

Throughout the paper we will use the following notation. A finite dimensional algebra over a field K will be always denoted by A, and its radical by N. By an A-module we will understand a right finite-dimensional A-module. A *complete ordered set of orthogonal idempotents* is a sequence $(e_1, e_2, \ldots, e_m)$ of orthogonal idempotents such that e_iA and e_jA do not contain isomorphic direct summands and $A(e_1+\cdots+e_m)A = A$. A *refinement* of such a set is a sequence $(e_{11}, e_{12}, \ldots, e_{1q_1}, e_{21}, e_{22}, \ldots, e_{2q_2}, \ldots, e_{m1}, e_{m2}, \ldots, e_{mq_m})$ of orthogonal idempotents such that for every t, $1 \le t \le m$ we have $A(e_{t1}+e_{t2}+\cdots+e_{tq_t})A = Ae_tA$. A *primitive refinement* is the one in which all the idempotents are primitive. In this case the sum $q_1+q_2+\cdots+q_m$ is equal to the number of isomorphism classes of simple A-modules, which will be denoted by n. Given an idempotent e of A, we call an idempotent ε an *orthogonal complement* of e in A if eA and εA do not contain isomorphic direct summands and $A(e+\varepsilon)A = A$. Following an earlier notation, given a complete ordered set of orthogonal idempotents $(e_1, e_2, \ldots, e_m)$ of A, we define the complete sequence of idempotents $(\varepsilon_1, \varepsilon_2 \ldots, \varepsilon_m)$ by $\varepsilon_t = e_t+e_{t+1}+\cdots+e_m$ for $t = 1, 2, \ldots, m$. For convenience we define $\varepsilon_{m+1} = 0$. If e is an idempotent element, $S(e)$ will stand for the semisimple right A-module eA/eN. The global dimension of A will be denoted by *gl dim* A, while *pd* M_A will stand for the projective dimension of an A-module M. Finally, the additive category generated by the summands of a module M will be denoted by *add* M.

§1. Statements

First we need the following characterization of idempotents in a neat sequence.

Proposition 1. *Let e be an idempotent in A and let ε be an orthogonal complement of e. Denote by C the subalgebra $\varepsilon A\varepsilon$. Then the following statements are equivalent:*

(i) $Ext^i_A(S(e), S(e)) = 0$ *for every* $i \ge 1$*;*

(ii) in a minimal projective resolution of $(eN)_A$

$$\cdots \to P_2 \to P_1 \to eN \to 0,$$

$P_i \in$ *add* (εA) *for all* $i \ge 1$*;*

(iii) the following properties hold:

(a) the multiplication map $eA\varepsilon \underset{C}{\otimes} \varepsilon Ae \to eA\varepsilon Ae$ *is bijective;*

(b) $eA\varepsilon Ae = eNe$*;*

(c) $Tor_i^C(eA\varepsilon, \varepsilon Ae) = 0$ *for every* $i \geq 1$.

Remark. It is easy to see that the conditions (a) and (c) are equivalent to the following ones:

(a') *the multiplication map* $A\varepsilon \otimes_C \varepsilon A \to A\varepsilon A$ *is bijective;*

(c') $Tor_i^C(A\varepsilon, \varepsilon A) = 0$ *for every* $i > 0$.

Proof. (i) $\Leftrightarrow$ (ii): Take a minimal projective resolution

$$\cdots \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} (S(e)) \longrightarrow 0$$

of the semisimple module $S(e)$. It is well known that $Ext_A^i(S(e), S(e)) \cong Hom_A(P_i, S(e))$. So the condition that the i-th extension group is 0 is equivalent to the condition that P_i does not contain any summands from *add* eA. Since $Ker\ \varphi_0 \cong eN$, we get the required equivalence.

(ii) $\Rightarrow$ (iii): Let us look at the minimal projective resolution

$$\cdots \to P_2 \to P_1 \to eN \to 0$$

of eN. Apply the functors $F = Hom_A(\varepsilon A, -)$ and $G = - \otimes_C \varepsilon A$ to get the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & GF(P_2) & \longrightarrow & GF(P_1) & \longrightarrow & GF(eN) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu} & & \\ \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & eN & \longrightarrow & 0, \end{array}$$

where μ_i and μ are the natural multiplication maps. Notice that the top row is exact. Indeed, since G is right exact and, by assumption, $P_i \in$ *add* εA, all μ_i $(i = 1, 2 \ldots)$ are isomorphisms, so the map $\mu :\ GF(eN) = eN\varepsilon \otimes_C \varepsilon A \to eN$ is also an isomorphism. Since $eN\varepsilon = eA\varepsilon$, the multiplication map $eA\varepsilon \otimes_C \varepsilon A \to eA\varepsilon A$ is bijective, $eA\varepsilon A = eN$, and also $Tor_i^C(eA\varepsilon, \varepsilon A) = 0$ for every $i \geq 1$. Thus the conditions of (iii) follow.

(iii) $\Rightarrow$ (ii): Now let us consider the minimal projective resolution

$$\cdots \to Q_2 \to Q_1 \to eN\varepsilon \to 0$$

of the C-module $eN\varepsilon = eA\varepsilon$. If we apply the functor G defined above, we get in view of our assumption the following exact sequence:

$$\cdots \to G(Q_2) \to G(Q_1) \to eN \to 0.$$

Here $G(Q_i) \in add\ \varepsilon A$, and thus we have (ii).

Definition: An idempotent e satisfying the conditions of Proposition 1 will be called a *neat idempotent.*

Obviously, this concept is two-sided. Also, a complete ordered set of orthogonal idempotents $(e_1, e_2, \dots, e_m)$ is a *neat sequence* if and only if e_i is neat in $\varepsilon_i A \varepsilon_i$ for every $i = 1, 2, \dots, m$.

Corollary 1. *If $(e_1, e_2, \dots, e_m)$ is a neat sequence, then it is also an* $\mathbf{r}$*-sequence.*

Remark. Condition (a) of (iii) also shows that neat algebras are constructed 'smoothly', in the sense of our introductory comments.

Corollary 2. *If e is a neat idempotent then there is a bijective correspondence between the members of the minimal projective resolutions of $(eN)_A$ and $(eN\varepsilon)_C = eA\varepsilon$; in particular: $pd\ (eN)_A = pd\ (eN\varepsilon)_C$.*

Let us recall the following well-known result:

Lemma. *Let $(e_1, e_2, \dots, e_n)$ be a complete ordered set of primitive orthogonal idempotents of A. Let $e = e_1 + \dots + e_k$ and $\varepsilon = e_{k+1} + \dots + e_n$ for some $1 \le k \le n$. Then the following are equivalent:*

(i) $eA\varepsilon Ae = eNe$;

(ii) $eN^2 e = eNe$;

(iii) the (valued oriented) graph of the algebra A has no arrows between any two vertices corresponding to the idempotents $e_1, e_2, \dots, e_k$.

Proof. One can use the identity $eN^2e = eN(e+\varepsilon)Ne = (eNe)^2 + eN\varepsilon Ne$ to prove the equivalence of (i) and (ii). The rest is now obvious.

Corollary 3. *If $(e_1, e_2, \dots, e_m)$ is an* $\mathbf{r}$*-sequence, then the graph of the algebra A has no loops. In particular the graph of a neat algebra has no loops.*

We are now ready to prove our basic statement.

Proposition 2. *Let e be a neat idempotent of A. Write $C = \varepsilon A\varepsilon$, where ε is an orthogonal complement of e. Then:*

(i) $gl\ dim\ C \leq gl\ dim\ A + pd\ (S(e))_A - 1 \leq 2gl\ dim\ A - 1$;

(ii) $gl\ dim\ A \leq gl\ dim\ C + pd\ (eA\varepsilon)_C + 2 \leq 2gl\ dim\ C + 2$.

Proof. (i) The statement is clear if $gl\ dim\ A = \infty$. So assume A is of finite global dimension. Let us consider the functor $F = Hom_A(\varepsilon A, -)$. It clearly carries the projective right A-module εA into a projective C-module. Since $F(eA) = F(eN)$ and by assumption, in the minimal projective resolution of the right A-module eN all the terms are in $add\ \varepsilon A$, we get that $pd\ (F(eA))_C \leq pd\ (eN)_A$. Thus the projective resolution

$$0 \to P_k \to \cdots \to P_0 \to S(\varepsilon) \to 0$$

is carried by the functor F to the resolution

$$0 \to F(P_k) \to \ldots \to F(P_0) \to F(S(\varepsilon)) \to 0;$$

here $pd\ (F(P_j))_C \leq pd(eN)_A$ for $j = 0, \ldots, k$. Since the simple C-modules are all in $add\ F(S(\varepsilon))$, the statement of (i) follows.

(ii) We may again assume that $gl\ dim\ C < \infty$. Using Corollary 2 we get that $(S(e))_A$ is either projective or $pd\ (S(e))_A = pd\ (eA\varepsilon)_C + 1$; by symmetry, if $(S(e))_A$ is not injective, then $id\ (S(e))_A = pd\ _C(\varepsilon Ae) + 1$. Let us denote the latter by k. From this it follows that if

$$\cdots \overset{\varphi_{i+1}}{\longrightarrow} P_i \overset{\varphi_i}{\longrightarrow} \cdots \overset{\varphi_1}{\longrightarrow} P_0 \overset{\varphi_0}{\longrightarrow} S(\varepsilon) \longrightarrow 0$$

is the minimal projective resolution of $S(\varepsilon)$, then $P_i \in add\ \varepsilon A$ for $i > k$; for $Ext^i_A(S(\varepsilon), S(e)) = 0$ in this case. Let us now denote with K_{k+1} the cokernel of the map $P_{k+2} \overset{\varphi_{k+2}}{\longrightarrow} P_{k+1}$. Thus we have the projective resolution

$$\cdots \overset{\varphi_{i+1}}{\longrightarrow} P_i \overset{\varphi_i}{\longrightarrow} \cdots \overset{\varphi_{k+2}}{\longrightarrow} P_{k+1} \overset{\varphi_{k+1}}{\longrightarrow} K_{k+1} \longrightarrow 0$$

of K_{k+1}. Here, by the choice of k, all the projective terms are in $add\ \varepsilon A$. Applying to this resolution the functors $F = Hom_A(\varepsilon A, -)$ and $G = - \underset{C}{\otimes} \varepsilon A$ we get the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & GF(P_{k+2}) & \longrightarrow & GF(P_{k+1}) & \longrightarrow & GF(K_{k+1}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_{k+2}} & & \downarrow{\scriptstyle \mu_{k+1}} & & \downarrow{\scriptstyle \mu} & & \\ \cdots & \longrightarrow & P_{k+2} & \longrightarrow & P_{k+1} & \longrightarrow & K_{k+1} & \longrightarrow & 0; \end{array}$$

clearly, μ_{k+2} and μ_{k+1} are isomorphisms, μ is a bijection and the top row is exact. This gives a correspondence between the projective resolution of $(K_{k+1})_A$ and that of $(F(K_{k+1}))_C$. Hence $pd\ (K_{k+1})_A \leq gl\ dim\ C$, and consequently $pd\ (S(\varepsilon))_A \leq gl\ dim\ C + pd\ _C(\varepsilon Ae) + 2$. By left-right symmetry we get the required inequality.

Applying Proposition 2 inductively, we can easily establish the theorem from the introduction.

Corollary 4. *If $(e_1, e_2, \ldots, e_m)$ is a neat sequence of idempotents in A, then $gl\ dim\ A \leq 2^m - 2$.*

2. Examples

Example 1. The following algebra shows that the bound of Proposition 2 (i) cannot be improved. Let A be the path algebra over the field K of the following graph:

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-1}} k \xrightarrow{\alpha_k} (k+1) \xrightarrow{\alpha_{k+1}} (k+2) \xrightarrow{\alpha_{k+2}} \cdots \xrightarrow{\alpha_{k+l}} (k+l+1)$$

modulo the ideal $< \alpha_t\alpha_{t+1} \mid 1 \leq t \leq k+l,\ t \neq k >$, where k and l are arbitrary natural numbers satisfying $2 \leq l \leq k$. Let us denote by e_i the idempotent corresponding to the vertex i. Then e_{k+1} is a neat idempotent, $pd\ S(e_1) = k$, $pd\ S(e_{k+1}) = l$, so $gl\ dim\ A = k$, while for $\varepsilon = 1 - e_{k+1}$ and $C = \varepsilon A \varepsilon$ we have $gl\ dim\ C = k + l - 1$. Thus the choice $k = l$ results in $gl\ dim\ C = 2gl\ dim\ A - 1$.

Example 2. The bound of Proposition 2 (ii) is also best possible. This time, let A be the path algebra of the graph

$$\begin{array}{ccc} & k+l & \\ \alpha_{k+l} \swarrow & & \nwarrow \alpha_{k+l-1} \\ 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-1}} k \xrightarrow{\alpha_k} & k+1 & \xrightarrow{\alpha_{k+1}} \cdots \xrightarrow{\alpha_{k+l-2}} k+l-1 \end{array}$$

modulo the ideal $< \alpha_{k+l}\alpha_1,\ \alpha_t\alpha_{t+1} \mid 1 \leq t \leq k+l-1,\ t \neq k >$, where k and l are arbitrary natural numbers satisfying $2 \leq k \leq l$. Then e_1 is a neat idempotent, $pd\ e_1 A \varepsilon_2 =$

$pd\ S(e_1) - 1 = k - 1$, and $pd\ S(e_{k+1}) = k + l$. So $gl\ dim\ A = k + l$, while $gl\ dim\ C = l - 1$, where $C = \varepsilon_2 A \varepsilon_2$. Again, the choice $k = l$ yields $gl\ dim\ A = 2gl\ dim\ C + 2$.

Example 3. Not every neat algebra is quasi-hereditary. The following algebra is neat but it has no **t**-sequences. Let us take the path-algebra of the graph:

$$\begin{array}{ccc} & 3 & \\ {}^{\gamma}\swarrow & & \nwarrow^{\beta} \\ 1 & \xrightarrow{\alpha} & 2 \end{array}$$

modulo the ideal $< \alpha\beta\gamma\alpha, \gamma\alpha\beta >$ (cf. [**DR2**]). Then the sequence (e_1, e_2, e_3) is clearly neat but the algebra is not quasi-hereditary.

Example 4. Not every algebra of finite global dimension is neat. If A is the path-algebra of the graph

$$1 \quad \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \\ \xleftarrow{\gamma} \end{array} \quad 2$$

modulo the ideal $< \beta\gamma, \gamma\alpha >$ (cf. [**G**]), then $gl\ dim\ A = 3$, but A is not neat.

Example 5. The following simple example shows that the existence of a **t**-sequence does not imply that the algebra has finite global dimension. Let us take for A the path algebra of the graph

$$1 \quad \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} \quad 2$$

modulo the ideal $< \alpha\beta\alpha, \beta\alpha\beta >$. Then (e_1, e_2) is a **t**-sequence, but $gl\ dim\ A = \infty$.

Example 6. The following example shows that the class of neat algebras is not closed under tilting. Let A be the path algebra of the graph

$$\begin{array}{ccc} & 1 & \\ {}^{\alpha}\nearrow & & \searrow^{\beta} \\ 2 & \begin{array}{c} \xleftarrow{\gamma} \\ \xrightarrow{\delta} \end{array} & 3 \end{array}$$

modulo the ideal $< \alpha\beta, \beta\gamma, \gamma\delta >$. It is easy to see, that A is neat but not quasi-hereditary. Let T_A be defined by $T_A = e_1A \oplus e_2A \oplus X$, where $X = (e_1A \oplus e_2A)/(\beta - \delta)A$. Then T is a tilting module, and the algebra $B = End(T_A)$ is not neat. Let us also notice that for the tilting module $T'_A = e_2A \oplus X \oplus Y$, where X is as before and $Y = e_2A/\delta A$, the algebra $B' = End(T'_A)$ is quasi-hereditary.

Now let us turn from examples to a simple application of these results. First we need the following criterion for **t**-sequences; the proof is very similar to that of Proposition 1.

Proposition 3. *Let ε be an idempotent of A, $C = \varepsilon A \varepsilon$ and X an arbitrary A-module. Then the following are equivalent:*

(i) the multiplication map $X\varepsilon \underset{C}{\otimes} \varepsilon A \to X\varepsilon A$ is bijective;

(ii) in the minimal projective resolution

$$\cdots \to P_1 \to P_0 \to X\varepsilon A \to 0$$

of $X\varepsilon A$ the projective modules P_0 and P_1 belong to $add\ \varepsilon A$.

Proof. (i) $\Rightarrow$ (ii): Assume first that the multiplication map $\mu : X\varepsilon \underset{C}{\otimes} \varepsilon A \to X\varepsilon A$ is bijective and let $Q_1 \to Q_0 \to (X\varepsilon)_C \to 0$ be the minimal projective resolution. Applying the right exact functor $G = - \underset{C}{\otimes} \varepsilon A$ to this sequence we obtain the exact sequence $G(Q_1) \to G(Q_0) \to G(X\varepsilon) \to 0$. By the assumption $G(X\varepsilon) = X\varepsilon \underset{C}{\otimes} \varepsilon A \cong X\varepsilon A$, and clearly $G(Q_i) = Q_i \underset{C}{\otimes} \varepsilon A$ are in $add\ \varepsilon A$ for $i = 0, 1$. The statement follows.

(ii) $\Rightarrow$ (i): Assume now that $P_1 \to P_0 \to X\varepsilon A \to 0$ is exact with $P_i \in add\ \varepsilon A$. Applying the functors $F = Hom_A(\varepsilon A, -)$ and $G = - \underset{C}{\otimes} \varepsilon A$ to this sequence, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} GF(P_1) & \longrightarrow & GF(P_0) & \longrightarrow & GF(X\varepsilon A) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \mu} & & \\ P_1 & \longrightarrow & P_0 & \longrightarrow & X\varepsilon A & \longrightarrow & 0, \end{array}$$

where μ_i and μ are the natural multiplication maps. Since by assumption μ_1 and μ_0 are bijective, so is $\mu : GF(X\varepsilon A) \cong X\varepsilon A\varepsilon \underset{C}{\otimes} \varepsilon A = X\varepsilon \underset{C}{\otimes} \varepsilon A \to X\varepsilon A$.

For the proof of the next result we refer the reader to **[DR1]**.

Proposition 4. *Let ε and ε' be idempotent elements in the algebra A such that $\varepsilon\varepsilon' = \varepsilon'\varepsilon = \varepsilon$. Write $B = A/A\varepsilon A$, $C = \varepsilon A\varepsilon$, $C' = \varepsilon' A\varepsilon'$, $\bar{C}' = \bar{\varepsilon}' B\bar{\varepsilon}'$, where $\bar{\varepsilon}' = \varepsilon' + A\varepsilon A$. Then the diagram*

$$\begin{array}{ccccccccc} & & A\varepsilon \underset{C}{\otimes} \varepsilon A & \longrightarrow & A\varepsilon' \underset{C'}{\otimes} \varepsilon' A & \longrightarrow & B\bar{\varepsilon}' \underset{\bar{C}'}{\otimes} \bar{\varepsilon}' B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \bar{\mu}'} & & \\ 0 & \longrightarrow & A\varepsilon A & \longrightarrow & A\varepsilon' A & \longrightarrow & B\bar{\varepsilon}' B & \longrightarrow & 0, \end{array}$$

where the rows are exact and the maps μ, μ' and $\bar{\mu}'$ are the natural multiplication maps, is commutative.

Corollary 5. *A sequence $(e_1, e_2, \ldots, e_m)$ is a* **t***-sequence if and only if for all $t, 1 \le t \le m$ the multiplication maps $B_t\bar{e}_t \underset{\bar{e}_t B_t \bar{e}_t}{\otimes} \bar{e}_t B_t \to B_t\bar{e}_t B_t$ (where $\bar{e}_t = e_t + A\varepsilon_{t+1}A$ in $B_t = A/A\varepsilon_{t+1}A$) are bijective.*

Let us now recall a lemma of Auslander (cf. [**A**]):

Lemma. *Let R be an algebra with radical denoted by N such that $N^d = 0$ but $N^{d-1} \neq 0$. Let X_R be an arbitrary R-module with $XN^m = 0$ for some $1 \le m \le d$. Then there is an exact sequence*

$$0 \to M_1 \to M_2 \to \cdots \to M_m \to X_R \to 0$$

such that:

(i) $M_i \in add\, (\overset{i}{\underset{t=1}{\oplus}} R/N^t)$;

(ii) *for any $Y_R \in add\ M_R$, the functor $Hom_R(Y, -)$ preserves the exactness of the above sequence.*

Using these preliminaries one gets easily the following result from [**DR3**]:

Proposition 5. *Let R be an algebra with radical N such that $N^d = 0$ but $N^{d-1} \neq 0$. Let $A = End_R(\overset{d}{\underset{t=1}{\oplus}} R/N^t)$, and let $(e_1, e_2, \ldots, e_d)$ be a complete ordered set of orthogonal idempotents of A such that $A\varepsilon_t A$ is the set of all endomorphisms factoring through $\overset{d-t+1}{\underset{k=1}{\oplus}} R/N^k$. Then $(e_1, e_2, \ldots, e_d)$ is a neat* **t***-sequence of A.*

Let us finally add one more remark. In [**Z**] Zacharia proved that the determinant of the Cartan matrix $C(A)$ of every algebra of global dimension 2 is $+1$. The idea of his proof can be extended in an obvious way to show that $det(C(A)) = +1$ for every neat algebra A.

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