HOMOLOGICAL CHARACTERIZATION OF LEAN ALGEBRAS

István Ágoston[1], Vlastimil Dlab[2] and Erzsébet Lukács[1]

Abstract. Certain classes of lean quasi-hereditary algebras play a central role in the representation theory of semisimple complex Lie algebras and algebraic groups. The concept of a lean semiprimary ring, introduced recently in [ADL] is given here a homological characterization in terms of the surjectivity of certain induced maps between Ext^1-groups. A stronger condition requiring the surjectivity of the induced maps between Ext^k-groups for all $k \geq 1$, which appears in the recent work of Cline, Parshall and Scott on Kazhdan–Lusztig theory, is shown to hold for a large class of lean quasi-hereditary algebras.

Throughout the paper R will denote a basic semiprimary ring with identity; thus the (Jacobson) radical J of R is nilpotent and R/J is a finite product of division rings. Let us fix a complete ordered set of primitive orthogonal idempotents $(e_1, e_2, \dots, e_n)$ and define for $1 \leq i \leq n$ the idempotent elements $\varepsilon_i = e_i + e_{i+1} + \dots + e_n$; set $\varepsilon_{n+1} = 0$. Thus, we have fixed an order on the set of the corresponding simple (right) R-modules $S(i)$ and their projective covers $P(i) \simeq e_iR$, $1 \leq i \leq n$. The corresponding left R-modules will be denoted by $S^\circ(i)$ and $P^\circ(i)$, respectively.

The *(right) standard modules* $\Delta(i)$ are defined by $\Delta(i) \simeq e_iR/e_iR\varepsilon_{i+1}R$. The submodule $e_iR\varepsilon_{i+1}R$ will be denoted by $V(i)$. Thus we have the exact sequence $0 \to V(i) \to P(i) \to \Delta(i) \to 0$. Similarly, we can define the *left standard modules* $\Delta^\circ(i)$ and the corresponding kernels $V^\circ(i)$.

The module $\Delta(i)$ is *Schurian* if $\mathrm{End}_R\big(\Delta(i)\big)$ is a division ring. It is easy to see that $\Delta(i)$ is Schurian if and only if $\Delta^\circ(i)$ is Schurian.

The ring R is *quasi-hereditary* (see [CPS]) with respect to the order $(e_1, e_2, \dots, e_n)$ if $\Delta(i)$ is Schurian for every $1 \leq i \leq n$ and the regular module R_R has a filtration $R_R = X_1 \supseteq X_2 \supseteq \dots \supseteq X_\ell \supseteq X_{\ell+1} = 0$ such that every factor X_i/X_{i+1}, $1 \leq i \leq \ell$ is isomorphic to a standard module $\Delta(j)$ for some

1991 Mathematics Subject Classification. Primary 16E99, 16S99. Secondary 17B10

[1] Research partially supported by NSERC of Canada and by Hungarian National Foundation for Scientific Research grant no. T1903

[2] Research partially supported by NSERC of Canada

$1 \le j \le n$. For basic facts concerning quasi-hereditary algebras we refer the reader to [DR1] and [DR2].

Let us now recall the definition of a top embedding ([ADL]). Let X and Y be arbitrary (right) R-modules. An embedding $f: X \to Y$ is called a *top embedding* if it induces an embedding $\bar{f}: X/\operatorname{rad} X = \operatorname{top} X \to \operatorname{top} Y = Y/\operatorname{rad} Y$. In this case we write $X \stackrel{t}{\subseteq} Y$. Note that, for a submodule $X \subseteq Y$, the condition $X \stackrel{t}{\subseteq} Y$ is equivalent to $\operatorname{rad} X = \operatorname{rad} Y \cap X$. A filtration $X = X_1 \supseteq X_2 \supseteq \ldots \supseteq X_m \supseteq X_{m+1} = 0$ of a module X is called a *top filtration* of X if $X_i \stackrel{t}{\subseteq} X$ for every $2 \le i \le m$. If $\mathcal{M}$ is a class of modules, then we will say that X has a *top filtration by* $\mathcal{M}$ if X has a top filtration $X = X_1 \supseteq X_2 \supseteq \ldots \supseteq X_m \supseteq X_{m+1} = 0$ such that the factor modules X_i/X_{i+1} belong to $\mathcal{M}$ for $1 \le i \le m$.

The semiprimary ring R is called *lean* with respect to the order $(e_1, e_2, \ldots, e_n)$ if $e_i J^2 e_j \subseteq e_i J \varepsilon_m J e_j$ for $m = \min\{i, j\}$ and $1 \le i, j \le n$. Theorem 2.1 of [ADL] asserts that A is lean if and only if $V(i) \stackrel{t}{\subseteq} \operatorname{rad} P(i)$ and $V^\circ(i) \stackrel{t}{\subseteq} \operatorname{rad} P^\circ(i)$ for all $1 \le i \le n$.

Lemma 1. *Let X be an arbitrary R-module and S a semisimple submodule of* $\operatorname{rad} X$. *Denote by Y the factor module X/S. Then the following statements are equivalent:*

(a) $S \stackrel{t}{\subseteq} \operatorname{rad} X$;

(b) *there exists an extension $\zeta \in \operatorname{Ext}^1(\operatorname{top} Y, S)$ such that the following diagram is commutative:*

$$\begin{array}{ccccccccc}
\zeta^\ell : 0 & \longrightarrow & S & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
\zeta : 0 & \longrightarrow & S & \longrightarrow & X' & \longrightarrow & \operatorname{top} Y & \longrightarrow & 0;
\end{array}$$

(c) *there exists a semisimple module T and an extension $\rho \in \operatorname{Ext}^1(T, S)$ such that the following diagram is commutative:*

$$\begin{array}{ccccccccc}
\rho^\ell : 0 & \longrightarrow & S & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \\
\rho : 0 & \longrightarrow & S & \stackrel{\iota}{\longrightarrow} & X'' & \longrightarrow & T & \longrightarrow & 0.
\end{array}$$

Proof. To prove $(a) \Rightarrow (b)$, observe that since S is semisimple, $S \stackrel{t}{\subseteq} \operatorname{rad} X$ implies that S is a direct summand of $\operatorname{rad} X$. Let C be a direct complement of S in $\operatorname{rad} X$. Then we have the following diagram with the natural maps:

$$\begin{array}{ccccccccc}
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S & \longrightarrow & X/C & \longrightarrow & Y' & \longrightarrow & 0.
\end{array}$$

Note that $Y' \simeq X/(C \oplus S) = X/\operatorname{rad} X = \operatorname{top} X \simeq \operatorname{top} Y$.

Since the implication $(b) \Rightarrow (c)$ is trivial, we have to show only that $(c) \Rightarrow (a)$. We need that $XJ^2 \cap S = 0$. Let us assume that $0 \neq S' = XJ^2 \cap S$. Then $0 \neq \iota(S') = \varphi(S') \subseteq \varphi(XJ^2) = \varphi(X)J^2$. But $\varphi(X) \subseteq X''$ and $X''J^2 = 0$, a contradiction. Thus $S \overset{t}{\subseteq} \operatorname{rad} X$. □

Proposition 2. *Let P_R be an indecomposable projective R-module and $V \subseteq \operatorname{rad} P$. Denote by W the factor module P/V. Then the following are equivalent:*

(a) $V \overset{t}{\subseteq} \operatorname{rad} P$;

(b) $\operatorname{Ext}^1(\operatorname{top} W, S) \to \operatorname{Ext}^1(W, S)$ *is an epimorphism for every simple module S.*

Proof. $(a) \Rightarrow (b)$ Consider a non-split exact sequence $0 \to S \to X \to W \to 0$; thus $S \subseteq \operatorname{rad} X$. Using the projectivity of P we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & V & \to & P & \to & W & \to & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \| & & \\ 0 & \to & S & \to & X & \to & W & \to & 0\,. \end{array}$$

Here ψ is an epimorphism, since $S \subseteq \operatorname{rad} X$. It follows that φ is also an epimorphism. We get that $S \overset{t}{\subseteq} \operatorname{rad} X$ since $V \overset{t}{\subseteq} \operatorname{rad} P$ by assumption. Thus, by Lemma 1, the sequence $0 \to S \to X \to W \to 0$ is a lifting of a sequence $0 \to S \to X' \to \operatorname{top} W \to 0$ along the natural map $W \to \operatorname{top} W$, so it is in the image of $\operatorname{Ext}^1(\operatorname{top} W, S) \to \operatorname{Ext}^1(W, S)$.

$(b) \Rightarrow (a)$ To prove that $V \overset{t}{\subseteq} \operatorname{rad} P$, it is sufficient to show that $V/V' \overset{t}{\subseteq} \operatorname{rad} P/V'$ for an arbitrary maximal submodule V' of the module V. Hence consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & V & \to & P & \to & W & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & V/V' & \to & P/V' & \to & W & \to & 0\,. \end{array}$$

Since V/V' is simple, (b) implies that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & V/V' & \to & P/V' & \to & W & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & V/V' & \to & Z & \to & \operatorname{top} W & \to & 0\,. \end{array}$$

By Lemma 1, we get that $V/V' \overset{t}{\subseteq} \operatorname{rad} P/V'$. □

Using Proposition 2 and the left dual version of it, we get immediately the following characterization of lean semiprimary rings.

THEOREM 3. *Let $(e_1, e_2, \dots, e_n)$ be a complete set of primitive orthogonal idempotents of the semiprimary ring R and let all standard modules $\Delta(i)$ be Schurian. Then R is lean with respect to the given order of idempotents if and only if the natural maps $\operatorname{Ext}^1\big(S(i), S(j)\big) \to \operatorname{Ext}^1\big(\Delta(i), S(j)\big)$ and $\operatorname{Ext}^1\big(S^\circ(i), S^\circ(j)\big) \to \operatorname{Ext}^1\big(\Delta^\circ(i), S^\circ(j)\big)$ are epimorphisms for all $1 \le i, j \le n$.*

Proof. Proposition 2 implies that the surjectivity of the maps given above is equivalent to the condition that $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$ and $V^\circ(i) \overset{t}{\subseteq} \operatorname{rad} P^\circ(i)$ for all $1 \le i \le n$. In turn, by Theorem 2.1 of [ADL], this is equivalent to the fact that R is lean. □

In what follows, let us restrict our attention to the case when $R = A$ is a finite dimensional K-algebra, where K is a field. For every $1 \le i \le n$, denote by $\nabla(i)$ the K-dual of $\Delta^\circ(i)$, and call the modules $\nabla(i)$ the *(right) costandard modules*. Using this terminology, we get the following characterization of lean quasi-hereditary K-algebras.

COROLLARY 4. *Let A be a quasi-hereditary K-algebra with respect to the order $(e_1, e_2, \dots, e_n)$. Then A is lean with respect to the same order if and only if the natural maps $\operatorname{Ext}^1\big(S(i), S(j)\big) \to \operatorname{Ext}^1\big(\Delta(i), S(j)\big)$ and $\operatorname{Ext}^1\big(S(j), S(i)\big) \to \operatorname{Ext}^1\big(S(j), \nabla(i)\big)$ are epimorphisms for $1 \le i, j \le n$.*

In their contributions to the Workshop on Representation Theory held in Ottawa in August 1992, B.J. Parshall and L.L. Scott emphasized the importance of the surjectivity of all natural maps $\operatorname{Ext}^k\big(S(i), S(j)\big) \to \operatorname{Ext}^k\big(\Delta(i), S(j)\big)$, $k \ge 1$, for the Kazhdan–Lusztig theory. In this connection, the following theorem and its corollary seem to be of some interest.

THEOREM 5. *Let A be a quasi-hereditary K-algebra with respect to the order $(e_1, e_2, \dots, e_n)$ such that $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$ for $1 \le i \le n$. Suppose that for every $1 \le i \le n$, the module $V(i)$ has a top filtration by $\Delta(j)$'s and $P(j)$'s, $i+1 \le j \le n$. Then the natural maps $\operatorname{Ext}^k\big(S(i), S(j)\big) \to \operatorname{Ext}^k\big(\Delta(i), S(j)\big)$ are surjective for all $1 \le i, j \le n$ and $k \ge 1$.*

For the proof of Theorem 5 we shall need the following simple lemma.

LEMMA 6. *Let $0 \to X \overset{\mu}{\to} Y \to Z \to 0$ be a short exact sequence with a top embedding μ. If, for a module S and for some $k \ge 1$, the natural maps $\operatorname{Ext}^k(\operatorname{top} X, S) \to \operatorname{Ext}^k(X, S)$ and $\operatorname{Ext}^k(\operatorname{top} Z, S) \to \operatorname{Ext}^k(Z, S)$ are surjective, then so is the natural map $\operatorname{Ext}^k(\operatorname{top} Y, S) \to \operatorname{Ext}^k(Y, S)$.*

Proof. The bottom sequence of the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X & \to & Y & \to & Z & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \operatorname{top} X & \to & \operatorname{top} Y & \to & \operatorname{top} Z & \to & 0 \end{array}$$

clearly splits. Thus, applying the functor $\operatorname{Hom}(-, S)$, we can derive easily the following commutative diagram from the long exact sequences:

$$\begin{array}{ccccccccc} \operatorname{Ext}^{k-1}(X,S) & \to & \operatorname{Ext}^k(Z,S) & \to & \operatorname{Ext}^k(Y,S) & \to & \operatorname{Ext}^k(X,S) & \to & \operatorname{Ext}^{k+1}(Z,S) \\ \uparrow & & \uparrow{\scriptstyle\gamma} & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\alpha} & & \uparrow \\ 0 & \to & \operatorname{Ext}^k(\operatorname{top} Z,S) & \to & \operatorname{Ext}^k(\operatorname{top} Y,S) & \to & \operatorname{Ext}^k(\operatorname{top} X,S) & \to & 0\,. \end{array}$$

Since α and γ are surjective, we get that β is surjective as well. □

Proof of Theorem 5. We proceed by induction. Proposition 2 implies that the statement holds for $k = 1$. Thus assuming the statement for some $k \geq 1$, we want to show that for every exact sequence

$$(*) \qquad 0 \to S(j) \to X_1 \to \ldots \to X_k \to X_{k+1} \to \Delta(i) \to 0$$

there is a commutative diagram of exact sequences with the natural projection $\Delta(i) \to S(i)$:

$$\begin{array}{ccccccccccccccc} 0 & \to & S(j) & \to & Y_1 & \to & \ldots & \to & Y_k & \to & P(i) & \to & \Delta(i) & \to & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S(j) & \to & Z_1 & \to & \ldots & \to & Z_k & \to & Z_{k+1} & \to & S(i) & \to & 0\,, \end{array}$$

in which the first row is equivalent to $(*)$.

Let us write $(*)$ as the Yoneda composite of the following exact sequences:

$$0 \to S(j) \to X_1 \to \ldots \to X_k \to N \to 0 \quad \text{and} \quad 0 \to N \to X_{k+1} \to \Delta(i) \to 0\,.$$

In view of the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \to & V(i) & \to & P(i) & \to & \Delta(i) & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & N & \to & X_{k+1} & \to & \Delta(i) & \to & 0 \end{array}$$

and

$$\begin{array}{ccccccccccccc} 0 & \to & S(j) & \to & Y_1 & \to & \ldots & \to & Y_k & \to & V(i) & \to & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \to & S(j) & \to & X_1 & \to & \ldots & \to & X_k & \to & N & \to & 0 \end{array}$$

the sequence $(*)$ is equivalent to

$$0 \to S(j) \to Y_1 \to \ldots \to Y_k \to P(i) \to \Delta(i) \to 0\,.$$

Now, by the induction hypothesis and by repeated use of Lemma 6, we get a commutative diagram of exact sequences

$$\begin{array}{ccccccccccccc} 0 & \to & S(j) & \to & Y_1 & \to & \ldots & \to & Y_k & \to & V(i) & \to & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \to & S(j) & \to & Z_1 & \to & \ldots & \to & Z_k & \to & \operatorname{top} V(i) & \to & 0\,. \end{array}$$

Furthermore, in view of Proposition 2, there is a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & V(i) & \to & P(i) & \to & \Delta(i) & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \operatorname{top} V(i) & \to & Z & \to & \Delta(i) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \operatorname{top} V(i) & \to & Z_{k+1} & \to & S(i) & \to & 0\,. \end{array}$$

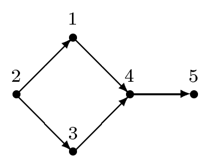
Hence the theorem follows. □

Corollary 7. *Let A be a shallow, medial or replete quasi-hereditary algebra with respect to $(e_1, e_2, \ldots, e_n)$. Then the natural maps* $\operatorname{Ext}^k\big(S(i), S(j)\big) \to \operatorname{Ext}^k\big(\Delta(i), S(j)\big)$ *and* $\operatorname{Ext}^k\big(S^\circ(i), S^\circ(j)\big) \to \operatorname{Ext}^k\big(\Delta^\circ(i), S^\circ(j)\big)$ *are surjective for all $1 \le i, j \le n$ and $k \ge 1$.*

The definition of shallow, right medial, left medial and replete algebras can be found in [ADL]. For the convenience of the reader, we wish to recall that these algebras are defined by the fact that $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$, $V^\circ(i) \overset{t}{\subseteq} \operatorname{rad} P^\circ(i)$ and, respectively, $V(i)$ and $V^\circ(i)$ have top filtrations by $\Delta(j)$'s and $\Delta^\circ(j)$'s, by $\Delta(j)$'s and $P^\circ(j)$'s, by $P(j)$'s and $\Delta^\circ(j)$'s and, finally, by $P(j)$'s and $P^\circ(j)$'s.

Remark 8. Let us point out that, in general, lean quasi-hereditary algebras do not satisfy the above surjectivity conditions for higher Ext-groups. Here is a simple example.

Let A be the path algebra of the graph



modulo the relations $\alpha_{14}\alpha_{45} = 0$ and $\alpha_{21}\alpha_{14} = \alpha_{23}\alpha_{34}$ (where α_{ij} denotes the arrow from i to j). Thus the right regular representation of A can be described by the following charts of composition factors:

$$A_A = \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \end{smallmatrix} \; .$$

One can check easily that A is lean. On the other hand $\mathrm{Ext}^2\big(S(2), S(5)\big) = 0$, while $\mathrm{Ext}^2\big(\Delta(2), S(5)\big) \neq 0$.

References

[ADL] Ágoston, I., Dlab, V., Lukács, E., Lean quasi-hereditary algebras, *Can. Math. Soc. Conference Proceedings Series* Vol. **13** (1993), 1–14.

[CPS] Cline, E., Parshall, B.J., Scott, L.L., Finite dimensional algebras and highest weight categories, *J. Reine Angew. Math.* **391** (1988), 85–99.

[DR1] Dlab, V., Ringel, C.M., Quasi-hereditary algebras, *Illinois J. of Math.* **35** (1989), 280–291.

[DR2] Dlab, V., Ringel, C.M., The module theoretical approach to quasi-hereditary algebras, *Representations of Algebras and Related Topics*, London Math. Soc. Lecture Note Series **168**, Cambridge Univ. Press 1992, 200–224.

Mathematical Institute of the Hungarian Academy of Sciences, P.O.Box 127, 1364 Budapest, Hungary
E-mail address: h4134ago@ella.hu

Department of Mathematics and Statistics, Carleton University, Ottawa, Ontario, K1S 5B6, Canada
E-mail address: vdlab@ccs.carleton.ca

Department of Mathematics, Faculty of Transport Engineering, Technical University of Budapest, 1111 Budapest, Hungary
E-mail address: h4091luk@ella.hu