LEAN QUASI-HEREDITARY ALGEBRAS

István Ágoston[1], Vlastimil Dlab[2] and Erzsébet Lukács[1]

Abstract. A class of quasi-hereditary algebras, related to a variety of applications, is introduced and studied in the paper. We call these algebras lean; they are characterized by the property that the species of consecutive centralizer algebras of projective modules, defined by a heredity sequence of idempotents, can be obtained by restrictions. Lean algebras are also characterized in terms of the so-called top filtrations (of the radical of the algebra). Furthermore, some canonical constructions of lean algebras are given for any ordered species.

In connection with their studies of highest weight categories arising in the representation theory of complex semisimple Lie algebras and algebraic groups, Cline, Parshall and Scott introduced the notion of a quasi-hereditary algebra ([CPS1], [PS]). This concept, defined purely in ring theoretical terms, has shortly proved to play an important role in a number of applications. Notably, the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ ([BGG]) has been shown to be a categorical sum of blocks which are equivalent to module categories over quasi-hereditary algebras (see e.g. [S]). Recent work of Cline, Parshall and Scott ([CPS2]), Beilinson, Ginsburg and Soergel ([BGS]) and Dyer ([D]) indicate that the quasi-hereditary algebras which appear in the respective applications are of a very particular type. The present paper represents an attempt to describe a class of quasi-hereditary algebras with additional properties in terms of the so-called top filtrations which may prove to be of importance in this connection. These algebras have a particular affinity to the $A(\gamma)$ construction of [DR2]: the species of the centralizers of projective modules (in a prescribed order) are obtained by the successive restrictions. In a separate paper, we shall provide a homological characterization of this class which will lead to a study of the relevant formal (quadratic) algebras as defined by Beilinson and Ginsburg ([BG]).

1991 Mathematics Subject Classification. Primary 16D99, 16P99, 16S99

[1] Research partially supported by NSERC of Canada and by Hungarian National Foundation for Scientific Research grant no. T4265

[2] Research partially supported by NSERC of Canada

1. Preliminaries. Top filtrations

Let R be a semiprimary ring with identity; thus the (Jacobson) radical J of R is nilpotent and R/J is Artinian. For simplicity we will assume that R is basic, i.e. R/J is the product of division rings. If $\{ e_1, e_2, \dots, e_n \}$ is a complete set of primitive orthogonal idempotents, then we shall usually fix an order of the idempotents $(e_1, e_2, \dots, e_n)$. For a given order of the primitive idempotents we shall define the idempotent elements $\varepsilon_i = e_i + e_{i+1} + \ldots + e_n$ for $1 \le i \le n$; let $\varepsilon_{n+1} = 0$. The principal indecomposable projective (right) module isomorphic to e_iR will be denoted by $P(i)$ and its simple top by $S(i)$. The corresponding left modules will be $P^\circ(i)$ and $S^\circ(i)$. Clearly, $\varepsilon_i R \simeq \bigoplus_{j=i}^{n} P(j)$ for $1 \le i \le n$.

The *species* of R is defined as $\mathcal{S}(R) = (D_1, D_2, \dots, D_n\, ;\, {}_iW_j\, ,\ 1 \le i,j \le n)$, where $D_i = e_iRe_i/e_iJe_i$ and ${}_iW_j = e_iJe_j/e_iJ^2e_j$. Thus, if $R = A$ is a finite dimensional K-algebra over a central field K then D_i is a finite dimensional division ring over K and ${}_iW_j$ a D_i-D_j-bimodule with K acting centrally for $1 \le i,j \le n$. When we want to stress the order of the idempotents, we will speak about an *ordered species*.

The *trace* $\tau_M X$ of a module M on a module X is defined as the submodule of X spanned by all submodules which are homomorphic images of M: $\tau_M X = \sum \{ \operatorname{Im} f \mid f \in \operatorname{Hom}(M,X) \}$. Thus, given an order $(e_1, e_2, \dots, e_n)$, every module has a filtration obtained by taking the traces of the projective R-modules $\varepsilon_i R$, $1 \le i \le n$ on X:

$$X = \tau_{\varepsilon_1 R} X \supseteq \tau_{\varepsilon_2 R} X \supseteq \ldots \supseteq \tau_{\varepsilon_i R} X \supseteq \ldots \supseteq \tau_{\varepsilon_n R} X \supseteq \tau_{\varepsilon_{(n+1)} R} X = 0\,.$$

Denoting by $X^{(i)}$ the trace $\tau_{\varepsilon_i R} X$, we can see easily that $X^{(i)} = X\varepsilon_i R$. We shall call this the *trace filtration* of X (with respect to a given order). In particular, for $X = R_R$ we get in this way a chain of two-sided idempotent ideals $I_i = R^{(i)} = R\varepsilon_i R$.

Of particular interest are the *(right) standard modules* $\Delta(i)$. For $1 \le i \le n$, the module $\Delta(i)$ is the first non-zero factor in the trace filtration of $P(i)$: $\Delta(i) = P(i)/\big(P(i)^{(i+1)}\big)$. Similarly, we can define the *left standard modules* $\Delta^\circ(i)$. Thus we can write the following exact sequences:

$$0 \to V(i) \to P(i) \to \Delta(i) \to 0\,,$$
$$0 \to U(i) \to \Delta(i) \to S(i) \to 0\,,$$
$$0 \to V(i) \to \operatorname{rad} P(i) \to U(i) \to 0\,.$$

Hence $V(i) \simeq P(i)^{(i+1)}$ and $U(i) \simeq \operatorname{rad} \Delta(i)$. Of course, there are similar sequences for the left modules $V^\circ(i), P^\circ(i), \Delta^\circ(i), U^\circ(i)$ and $S^\circ(i)$.

In case $R = A$ is a finite dimensional K-algebra, where K is a central field, the K-dual of $\Delta^\circ(i)$, $1 \le i \le n$ will be denoted by $\nabla(i)$ and they will be referred to as the *(right) costandard modules*. Clearly, $\Delta(i)$ is the largest factor module of $P(i)$ such that the composition factors are all isomorphic to some $S(j)$ for $j \le i$; and dually, $\nabla(i)$ is the largest submodule of $Q(i)$, the injective hull of $S(i)$, for which the composition factors are isomorphic to some $S(j)$ for $j \le i$. We should also note that the sequence $\Delta = \big(\Delta(1), \Delta(2), \dots, \Delta(n)\big)$ depends on the choice of the order $(e_1, e_2, \dots, e_n)$.

We shall call the module $\Delta(i)$ *Schurian* if $\operatorname{End}_R\big(\Delta(i)\big)$ is a division ring. It is easy to see that $\Delta(i)$ is Schurian if and only if $\Delta^\circ(i)$ is Schurian. The sequence Δ is *Schurian* if every $\Delta(i)$ is Schurian for $1 \le i \le n$. Note that $\Delta(i)$ is Schurian if and only if $S(i)$ does not appear as a composition factor of $\operatorname{rad}\Delta(i)$.

Recall that the ring R is called *quasi-hereditary* with respect to the order $(e_1, e_2, \dots, e_n)$ if Δ is Schurian and the factor modules I_i/I_{i+1} for $1 \le i \le n$ from the trace filtration of R_R are direct sums of $\Delta(i)$'s (or equivalently, the same conditions hold for left modules). For basic properties of quasi-hereditary algebras we refer to [DR1] and [DR5].

An embedding of a module X into a module Y will be called a *top embedding* if it induces an embedding of $\operatorname{top} X = X/\operatorname{rad} X$ into $\operatorname{top} Y = Y/\operatorname{rad} Y$. In this case we shall write $X \overset{t}{\subseteq} Y$. Clearly, for $X \subseteq Y$ the condition $X \overset{t}{\subseteq} Y$ is equivalent to $\operatorname{rad} X = \operatorname{rad} Y \cap X$, or in fact to $\operatorname{rad} X \supseteq \operatorname{rad} Y \cap X$.

LEMMA 1.1. *Let $X \subseteq Y \subseteq Z$ be R-modules.*
(a) *$X \overset{t}{\subseteq} Z$ implies $X \overset{t}{\subseteq} Y$.*
(b) *$X \overset{t}{\subseteq} Y$ and $Y \overset{t}{\subseteq} Z$ implies $X \overset{t}{\subseteq} Z$.*
(c) *Suppose $X \overset{t}{\subseteq} Z$. Then $Y/X \overset{t}{\subseteq} Z/X$ if and only if $Y \overset{t}{\subseteq} Z$.*

Proof. We shall prove only (c) as the first two statements are straightforward. Then we have the following sequence of equivalent conditions:

$$Y/X \overset{t}{\subseteq} Z/X$$
$$(Y \cap \operatorname{rad} Z) + X \subseteq \operatorname{rad} Y + X$$
$$Y \cap \operatorname{rad} Z = (Y \cap \operatorname{rad} Z) \cap (\operatorname{rad} Y + X).$$

But $Y \cap \operatorname{rad} Z \cap (\operatorname{rad} Y + X) = Y \cap \big(\operatorname{rad} Y + (\operatorname{rad} Z \cap X)\big) = Y \cap (\operatorname{rad} Y + \operatorname{rad} X) = Y \cap \operatorname{rad} Y = \operatorname{rad} Y$. Thus the initial condition is equivalent to $Y \cap \operatorname{rad} Z = \operatorname{rad} Y$, that is to $Y \overset{t}{\subseteq} Z$. □

A filtration $X = X_1 \supseteq X_2 \supseteq \ldots \supseteq X_m \supseteq X_{m+1} = 0$ of a module X is called a *top filtration* of X if $X_i \overset{t}{\subseteq} X$ for every $2 \le i \le m$. In view of Lemma 1.1, this is equivalent to the condition that $X_i/X_{i+1} \overset{t}{\subseteq} X/X_{i+1}$ for every $2 \le i \le m$. It also follows that any given top filtration of a module X can be refined to a top filtration where the factors of consecutive terms all have simple tops.

Our interest will be directed mainly towards those modules where the trace filtration (or a part of it) is a top filtration.

2. Lean semiprimary rings

Let R be a basic semiprimary ring with radical J and let $(e_1, e_2, \ldots, e_n)$ be a complete ordered set of primitive orthogonal idempotents. We say that R is *lean* with respect to this order if $e_i J^2 e_j = e_i J \varepsilon_m J e_j$ for every $1 \le i, j \le n$, where $m = \min\{\, i, j \,\}$. R will be called *lean quasi-hereditary* if and only if R is lean and quasi-hereditary with respect to the same order.

Clearly, for R to be lean is equivalent to the condition that the species $\mathcal{S}(C_t)$ of the centralizer ring $C_t = \varepsilon_t R \varepsilon_t$ can be obtained from $\mathcal{S}(R)$, the (ordered) species of R, by restricting it to the indices $\{\, t, t+1, \ldots, n \,\}$. That is to say, if $\mathcal{S}(R) = (D_1, D_2, \ldots, D_n\, ;\, {}_iW_j\, ,\ 1 \le i, j \le n)$, then $\mathcal{S}(C_t) = (D_t, D_{t+1}, \ldots, D_n\, ;\, {}_iW_j\, ,\ t \le i, j \le n)$.

In general there are many (quasi-hereditary) algebras which are not lean. Consider for example the simple case of the (hereditary) path K-algebra A of the graph $\overset{2}{\bullet}\xrightarrow{\ \alpha\ }\overset{1}{\bullet}\xrightarrow{\ \beta\ }\overset{3}{\bullet}$; here the graph of the corresponding (hereditary) centralizer algebra C_2 is $\overset{2}{\bullet}\xrightarrow{\ \alpha\beta\ }\overset{3}{\bullet}$, and thus not just the restriction of the original graph. Of course, this reflects the fact that $\alpha\beta \in e_2 J^2 e_3$ but $\alpha\beta \notin e_2 J \varepsilon_2 J e_3$. We should note, however, that in the case of hereditary algebras, an *admissible order* of the idempotents (i. e. where $e_i J e_j \ne 0$ implies $i < j$) always gives a lean order.

The following theorem formulates the relationship between lean rings and top trace filtrations.

THEOREM 2.1. *Let R be a semiprimary ring with radical J and $(e_1, e_2, \ldots, e_n)$ a complete ordered set of primitive orthogonal idempotents. Assume that the standard modules $\Delta(i)$ are Schurian for $1 \le i \le n$. Then the following conditions on R are equivalent:*

(1) *R is lean with respect to the given order, that is $e_i J^2 e_j = e_i J \varepsilon_m J e_j$ for $1 \le i, j \le n$ and $m = \min\{\, i, j \,\}$;*

(2) *the trace filtration of $U(i)$ is a top filtration and $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$ for $1 \le i \le n$;*

(2°) *the trace filtration of $U^\circ(i)$ is a top filtration and $V^\circ(i) \overset{t}{\subseteq} \operatorname{rad} P^\circ(i)$ for $1 \le i \le n$;*

(3) $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$ *and* $V^\circ(i) \overset{t}{\subseteq} \operatorname{rad} P^\circ(i)$ *for* $1 \le i \le n$*;*
(4) *the trace filtrations of* $U(i)$ *and of* $U^\circ(i)$ *are top filtrations for* $1 \le i \le n$*.*

To prove the theorem, we shall formulate first a few lemmas.

Lemma 2.2. *For a given sequence* $(e_1, e_2, \dots, e_n)$ *and an index* $1 \le i \le n$

$$V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$$

if and only if

$$e_i J^2 e_j = e_i J \varepsilon_{i+1} J e_j \quad \textit{for every} \quad j > i\,.$$

Proof. The proof follows from the following string of equivalent statements expressing the fact that $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$:

$$\begin{aligned} e_i J \varepsilon_{i+1} R &\overset{t}{\subseteq} e_i J \\ e_i J^2 \cap e_i J \varepsilon_{i+1} R &= e_i J \varepsilon_{i+1} J \\ e_i J^2 e_j \cap e_i J \varepsilon_{i+1} R e_j &= e_i J \varepsilon_{i+1} J e_j \quad \text{for all} \quad 1 \le j \le n\,. \end{aligned}$$

However, the last equality is trivial for $j \le i$, since $\varepsilon_{i+1} R e_j = \varepsilon_{i+1} J e_j$ and $J \varepsilon_{i+1} J \subseteq J^2$; moreover, for $j > i$, the left-hand side collapses to $e_i J^2 e_j$ since $J \varepsilon_{i+1} R e_j \supseteq J e_j \supseteq J^2 e_j$. □

Lemma 2.3. *For a given sequence* $(e_1, e_2, \dots, e_n)$ *and indices* $1 \le i, j \le n$

$$\big(\operatorname{rad} P^\circ(j)\big)^{(i)} / \big(\operatorname{rad} P^\circ(j)\big)^{(i+1)} \overset{t}{\subseteq} \operatorname{rad} P^\circ(j) / \big(\operatorname{rad} P^\circ(j)\big)^{(i+1)}$$

if and only if

$$e_i J^2 e_j = e_i J \varepsilon_i J e_j\,.$$

Proof. As in the proof of the previous lemma, we write down equivalent statements, expressing the top embedding from the lemma:

$$\begin{aligned} R\varepsilon_i J e_j / R\varepsilon_{i+1} J e_j &\overset{t}{\subseteq} J e_j / R\varepsilon_{i+1} J e_j\,, \\ R\varepsilon_i J e_j \cap (J^2 e_j + R\varepsilon_{i+1} J e_j) &= J \varepsilon_i J e_j + R\varepsilon_{i+1} J e_j\,, \\ e_k R\varepsilon_i J e_j \cap (e_k J^2 e_j + e_k R\varepsilon_{i+1} J e_j) &= \\ &= e_k J \varepsilon_i J e_j + e_k R \varepsilon_{i+1} J e_j \quad \text{for all} \quad 1 \le k \le n\,. \end{aligned}$$

For $k < i$, the last equality is trivial, since both sides equal $e_k J \varepsilon_i J e_j$. We can also verify easily that for $k > i$, both sides equal $e_k R \varepsilon_{i+1} J e_j$; just observe that $e_k R \varepsilon_i J \supseteq e_k R \varepsilon_{i+1} J \supseteq e_k R e_k J \supseteq e_k J^2$ and $e_k J \varepsilon_i J \subseteq e_k J \subseteq e_k R \varepsilon_{i+1} J$. Hence the only genuine condition remains for $k = i$: $e_i J^2 e_j = e_i J \varepsilon_i J e_j$. □

Using these lemmas we get the following proposition.

PROPOSITION 2.4. *Let R be a semiprimary ring with Schurian standard modules $\Delta(i)$. Then the following are equivalent:*

(a) $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$ *for* $1 \le i \le n$;

(b) $e_i J^2 e_j = e_i J \varepsilon_i J e_j$ *for* $1 \le i \le j \le n$;

(c) *the trace filtration of* $U^\circ(j)$ *is a top filtration for* $1 \le j \le n$.

Proof. Note that the fact that $\Delta(i)$ is Schurian means that $e_i J \varepsilon_i R = e_i J \varepsilon_{i+1} R$. In particular, this implies that the equation in (b) always holds for $i = j$. Now the equivalence of (a) and (b) follows from Lemma 2.2 and the Schurian property, while the equivalence of (b) and (c) is an immediate consequence of Lemma 2.3, since $\operatorname{rad} \Delta^\circ(j) / \big(\operatorname{rad} \Delta^\circ(j)\big)^{(i+1)} \simeq \operatorname{rad} P^\circ(j) / \big(\operatorname{rad} P^\circ(j)\big)^{(i+1)}$ for $i < j$. □

Of course, we can formulate also the dual of Proposition 2.4.

PROPOSITION 2.4°. *Let R be a semiprimary ring with Schurian standard modules $\Delta^\circ(i)$. Then the following are equivalent:*

(a°) $V^\circ(i) \overset{t}{\subseteq} \operatorname{rad} P^\circ(i)$ *for* $1 \le i \le n$;

(b°) $e_j J^2 e_i = e_j J \varepsilon_i J e_i$ *for* $1 \le i \le j \le n$;

(c°) *the trace filtration of* $U(j)$ *is a top filtration for* $1 \le j \le n$.

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We will use the conditions of Proposition 2.4 as well as those of its dual, Proposition 2.4°. Now, condition (1) of the theorem is equivalent to (b) and (b°); condition (2) is equivalent to (a) and (c°); condition (2°) to (a°) and (c); condition (3) to (a) and (a°); finally condition (4) to (c) and (c°). □

3. Special classes of lean quasi-hereditary algebras

From now on we will assume that $R = A$ is a finite dimensional algebra over a central field K. As before, $(e_1, e_2, \ldots, e_n)$ will be a fixed order of a set of primitive orthogonal idempotents and we shall assume that A is quasi-hereditary with respect to this order.

In this section we will describe two classes of lean algebras: shallow and replete, which in some sense lie in the opposite ends of the spectrum of lean quasi-hereditary algebras. We will also construct canonical shallow and replete algebras for any given species.

Recall ([DR4]) that a quasi-hereditary algebra A is called *shallow* with respect to the order $(e_1, e_2, \ldots, e_n)$ if the modules $U(i) = \operatorname{rad}\Delta(i)$ and $U^\circ(i) = \operatorname{rad}\Delta^\circ(i)$ are semisimple for $1 \le i \le n$.

It follows from condition (4) of Theorem 2.1 that shallow algebras are lean, since clearly, any filtration of a semisimple module is a top filtration. In fact, we have the following characterization of shallow algebras.

THEOREM 3.1. *Let A be a quasi-hereditary algebra with respect to an order $(e_1, e_2, \ldots, e_n)$. Then the following conditions on A are equivalent:*

(1) $e_iJ^2e_j = e_iJ\varepsilon_MJe_j$ *for every* $1 \le i, j \le n$, *where* $M = \max\{i, j\}$*;*

(2) *the trace filtration of* $\operatorname{rad}P(i)$ *is a top filtration and the consecutive factors* $\big(\operatorname{rad}P(i)\big)^{(j)}/\big(\operatorname{rad}P(i)\big)^{(j+1)}$ *are semisimple for* $1 \le j < i \le n$*;*

(2°) *the trace filtration of* $\operatorname{rad}P^\circ(i)$ *is a top filtration and the consecutive factors* $\big(\operatorname{rad}P^\circ(i)\big)^{(j)}/\big(\operatorname{rad}P^\circ(i)\big)^{(j+1)}$ *are semisimple for* $1 \le j < i \le n$*;*

(3) $V(i) \overset{t}{\subseteq} \operatorname{rad}P(i)$, $\;V^\circ(i) \overset{t}{\subseteq} \operatorname{rad}P^\circ(i)$, *and the trace filtrations of* $V(i)$ *and* $V^\circ(i)$ *are top filtrations for* $1 \le i \le n$*;*

(4) *A is shallow, i. e. $U(i)$ and $U^\circ(i)$ are semisimple for* $1 \le i \le n$.

Before proving the theorem we need the following observation.

LEMMA 3.2. *For a given sequence $(e_1, e_2, \ldots, e_n)$ and an index $1 \le i \le n$*

$$\Delta(i)J^2 = 0$$

if and only if

$$e_iJ^2e_j = e_iJ\varepsilon_{i+1}Je_j \quad \textit{for all} \quad j \le i\,.$$

Proof. Clearly, the condition $e_iJ^2e_j = e_iJ\varepsilon_{i+1}Je_j$ is equivalent to $e_iJ^2e_j \subseteq e_iJ\varepsilon_{i+1}Je_j$. From the definition of $\Delta(i)$ we get that $\Delta(i)J^2 = 0$ if and only if $e_iJ^2 \subseteq e_iJ\varepsilon_{i+1}A$ and this is equivalent to the condition that $e_iJ^2e_j \subseteq e_iJ\varepsilon_{i+1}Ae_j$ holds for every $1 \le j \le n$. But the last condition is always satisfied for $j > i$ since $e_iJ^2e_j \subseteq e_iJe_j \subseteq e_iJ\varepsilon_{i+1}Ae_j$. On the other hand, for $j \le i$ we have $e_iJ\varepsilon_{i+1}Ae_j = e_iJ\varepsilon_{i+1}Je_j$, hence the statement follows. □

PROPOSITION 3.3. *Let R be a semiprimary ring with Schurian standard modules $\Delta(i)$. Then the following are equivalent:*

(*a*) $\big(\operatorname{rad}P^\circ(j)\big)^{(i)}/\big(\operatorname{rad}P^\circ(j)\big)^{(i+1)} \overset{t}{\subseteq} \operatorname{rad}P^\circ(j)/\big(P^\circ(j)\big)^{(i+1)}$ *for* $1 \le j < i \le n$*;*

(*b*) $e_iJ^2e_j = e_iJ\varepsilon_iJe_j$ *for* $1 \le j \le i \le n$*;*

(*c*) $\Delta(i)J^2 = 0$ *for* $1 \le i \le n$.

Proof. The Schurian property of Δ implies that the equivalence of (a) and (b) follows from Lemma 2.3, while the equivalence of (b) and (c) is a consequence of Lemma 3.2. □

As in the case of Proposition 2.4, we can also formulate the dual statement Proposition 3.3° with conditions (a°), (b°) and (c°).

Proof of Theorem 3.1. We will use the conditions of Proposition 3.3 and its dual. Condition (1) of the theorem is equivalent to (b) and (b°); condition (2) to (a°) and (c); condition (2°) to (a) and (c°); condition (3) to (a) and (a°); finally conditon (4) to (c) and (c°). □

We have seen that shallow algebras are characterized by the fact that the radical $\operatorname{rad} P(i)$ of every principal indecomposable module has a top filtration with consecutive factors isomorphic to some $S(j)$ for $j < i$ or $\Delta(j)$ for $j > i$. Thus shallow algebras are "as small as possible" among (lean) quasi-hereditary algebras on a given species. The other extreme is realized by algebras which will be called replete. A quasi-hereditary algebra A is *replete* with respect to the order $(e_1, e_2, \dots, e_n)$ if the modules $U(i) = \operatorname{rad}\Delta(i)$ and $U^\circ(i) = \operatorname{rad}\Delta^\circ(i)$ have top trace filtrations with consecutive factors isomorphic to direct sums of $\Delta(j)$'s and $\Delta^\circ(j)$'s, respectively, for $1 \le j < i \le n$. Let us note that, in particular, replete algebras are lean. Moreover, it will be shown that replete algebras are "as big as possible" among lean (quasi-hereditary) algebras on a given species, since the radical $\operatorname{rad} P(i)$ of every principal indecomposable module has a top filtration with consecutive factors isomorphic to some $\Delta(j)$ for $j < i$ or $P(j)$ for $j > i$.

THEOREM 3.4. *Let A be a quasi-hereditary algebra with respect to an order $(e_1, e_2, \dots, e_n)$. Then the following conditions on A are equivalent:*

(1) *the trace filtration of $U(i)$ is a top filtration with the consecutive factors $\big(\operatorname{rad} P(i)\big)^{(j)}/\big(\operatorname{rad} P(i)\big)^{(j+1)}$ isomorphic to direct sums of $\Delta(j)$'s (where $j < i$), and $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$ with $V(i)$ projective for $1 \le i \le n$;*

(1°) *the trace filtration of $U^\circ(i)$ is a top filtration with the consecutive factors $\big(\operatorname{rad} P^\circ(i)\big)^{(j)}/\big(\operatorname{rad} P^\circ(i)\big)^{(j+1)}$ isomorphic to direct sums of $\Delta^\circ(j)$'s (where $j < i$), and $V^\circ(i) \overset{t}{\subseteq} \operatorname{rad} P^\circ(i)$ with $V^\circ(i)$ projective for $1 \le i \le n$;*

(2) *$V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$ and $V^\circ(i) \overset{t}{\subseteq} \operatorname{rad} P^\circ(i)$ with both $V(i)$ and $V^\circ(i)$ projective;*

(3) *A is replete, i. e. $U(i)$ and $U^\circ(i)$ have top filtrations with factors isomorphic to $\Delta(j)$ and $\Delta^\circ(j)$, respectively, for $1 \le j < i \le n$.*

The theorem is a consequence of the following proposition and its dual.

PROPOSITION 3.5 *Let A be a quasi-hereditary algebra with respect to an order $(e_1, e_2, \dots, e_n)$. Then the following conditions on A are equivalent:*
(i) $V^\circ(j)$ is projective for $1 \le j \le n$;
(ii) $U(i)$ has a Δ-filtration, i. e. a filtration where the factors of consecutive terms are all isomorphic to $\Delta(k)$'s for $1 \le k < i \le n$.

Proof. We can easily see that condition (i) can be reformulated as
(i') *proj.dim* $\Delta^\circ(j) \le 1$ *for* $1 \le j \le n$.
Also, in view of Theorem 1 of [DR5], for a quasi-hereditary algebra A condition (ii) is equivalent to
(ii') $\mathrm{Ext}^1\big(U(i), \nabla(j)\big) = 0$ *for* $1 \le i, j \le n$.
Consider now the exact sequence $0 \to U(i) \to \Delta(i) \to S(i) \to 0$. By applying the functor $\mathrm{Hom}\big(-, \nabla(j)\big)$ to the sequence and taking into the account that $\mathrm{Ext}^t\big(\Delta(i), \nabla(j)\big) = 0$ for $1 \le i, j \le n$ and $t \ge 1$ (cf. Theorem 1 of [DR5]), we get that $\mathrm{Ext}^1\big(U(i), \nabla(j)\big) \simeq \mathrm{Ext}^2\big(S(i), \nabla(j)\big)$. Now by using K-duality we get the following string of equivalent conditions:

$$\begin{aligned} \textit{proj.dim}\,\Delta^\circ(j) \le 1 \quad &\text{for} \quad 1 \le j \le n\,; \\ \mathrm{Ext}^2\big(\Delta^\circ(j), S^\circ(i)\big) = 0 \quad &\text{for} \quad 1 \le i, j \le n\,; \\ \mathrm{Ext}^2\big(S(i), \nabla(j)\big) = 0 \quad &\text{for} \quad 1 \le i, j \le n\,; \\ \mathrm{Ext}^1\big(U(i), \nabla(j)\big) = 0 \quad &\text{for} \quad 1 \le i, j \le n\,. \end{aligned}$$

This shows the equivalence of conditions (i') and (ii'). □

Proof of Theorem 3.4. Clearly, in view of Theorem 2.1, each of the conditions implies that A is lean. The rest will now follow from Proposition 3.4 (and its dual). Namely, the fact that $\mathrm{Ext}^1\big(\Delta(i), \Delta(j)\big) = 0$ for $1 \le j \le i \le n$ implies that the existence of any Δ-filtration for a module M is equivalent to the condition that the factors of the trace filtration are direct sums of $\Delta(i)$'s. □

COROLLARY 3.6. *The global dimension of a replete algebra is at most* 2.

Proof. This is a consequence of Theorem 3 of [DR3]. However, for the sake of completeness we give a short proof here.

We have seen that the projectivity of the modules $V(i)$ implies that *proj.dim* $\Delta(i) \le 1$ for $1 \le i \le n$. Thus from the existence of a Δ-filtration for $U(i)$ we also get that *proj.dim* $U(i) \le 1$ as well. Finally from the exact sequence

$$0 \to U(i) \to \Delta(i) \to S(i) \to 0$$

we obtain that *proj.dim* $S(i) \le 2$ for $1 \le i \le n$, as required. □

In view of the characterizations of shallow and replete algebras (Theorems 3.1 and 3.4), it is natural to define the following two "intermediate" classes of lean quasi-hereditary algebras. A quasi-hereditary algebra A is said to be *right medial* if all $\operatorname{rad} P(i)$ for $1 \le i \le n$ have top filtrations with factors isomorphic to some $\Delta(j)$ with $1 \le j \le n$, $i \ne j$. An algebra is said to be *left medial* if all $\operatorname{rad} P^\circ(i)$ for $1 \le i \le n$ have top filtrations with factors isomorphic to some $\Delta^\circ(j)$ with $1 \le j \le n$, $i \ne j$, i.e. if the opposite algebra A^o is right medial. Using Propositions 3.3 and 3.5 and the quasi-heredity of A we can see that this is equivalent to requiring that every $\operatorname{rad} P(i)$, $1 \le i \le n$ has top filtration with factors isomorphic to $S(j)$, $1 \le j < i$ and $P(j)$, $i < j \le n$. Other characterizations, similar to the ones given for shallow and replete algebras, can be given for these two classes, too.

4. Canonical constructions

In this final section we are going to construct the "canonical" shallow, medial and replete quasi-hereditary algebras over a given ordered species. Let $\mathcal{S} = (D_1, D_2, \dots, D_n\,;\, {}_iW_j\,,\ 1 \le i,j \le n)$ be an ordered species with ${}_iW_i = 0$ for all $1 \le i \le n$. Let $T(\mathcal{S})$ be the tensor algebra over $\mathcal{S}$:

$$T(\mathcal{S}) = \Lambda \oplus W \oplus W^{\otimes 2} \oplus W^{\otimes 3} \oplus \dots\,,$$

where $\Lambda = D_1 \times D_2 \times \dots \times D_n$, $W = \bigoplus_{i,j} {}_iW_j$ is a Λ-Λ-bimodule with Λ operating via the projections, all tensor products are over Λ and the multiplication is induced by $W^{\otimes r} \otimes_\Lambda W^{\otimes s} \simeq W^{\otimes r+s}$. Of course, $T(\mathcal{S})$ is, in general, infinite dimensional.

Define the following ideals in $T(\mathcal{S})$:

$$\begin{aligned} I_S &= \langle {}_iW_j \otimes {}_jW_k \mid j < \max\{\,i,k\,\}\rangle \\ I_{M_r} &= \langle {}_iW_j \otimes {}_jW_k \mid j < k\rangle \\ I_{M_\ell} &= \langle {}_iW_j \otimes {}_jW_k \mid i > j\rangle \quad \text{and} \\ I_R &= \langle {}_iW_j \otimes {}_jW_k \mid j < \min\{\,i,k\,\}\rangle \end{aligned}$$

Put

$$H(\mathcal{S}) = T(\mathcal{S})/I_H \text{ for } H = S, M_r, M_\ell, \text{ and } R\,.$$

We should mention here that the algebra $S(\mathcal{S})$ was already constructed in [DR4], while $R(\mathcal{S})$ was defined in [DR3] and called there a *peaked algebra* on $\mathcal{S}$.

THEOREM 4.1. *The algebras $S(\mathcal{S})$, $M_r(\mathcal{S})$, $M_\ell(\mathcal{S})$, and $R(\mathcal{S})$ are quasi-hereditary algebras with the ordered species $\mathcal{S}$. The algebra $S(\mathcal{S})$ is shallow, $M_r(\mathcal{S})$ right medial, $M_\ell(\mathcal{S})$ left medial, and $R(\mathcal{S})$ replete.*

In fact,

$$S(\mathcal{S}) \simeq \Lambda \oplus W \oplus \big(\bigoplus_{\substack{t>i \\ t>j}} {}_iW_t \otimes {}_tW_j \big) \, ,$$

and $M_r(\mathcal{S})$, $M_\ell(\mathcal{S})$ and $R(\mathcal{S})$ are isomorphic to

$$\Lambda \oplus W \oplus (\oplus \, {}_{i_0}W_{i_1} \otimes \ldots \otimes {}_{i_t}W_{i_{t+1}} \otimes \ldots \otimes {}_{i_{m-1}}W_{i_m}) \, ,$$

where the summation runs through all sequences $(i_0, i_1, \ldots, i_t, \ldots, i_{m-1}, i_m)$, $m \geq 2$, subject to

$$i_1 > i_2 > \ldots > i_m \, ,$$
$$i_0 < i_1 < \ldots < i_{m-1} \, , \quad \text{and}$$
$$i_0 < i_1 < \ldots < i_t > \ldots > i_{m-1} > i_m \, , \quad 0 \leq t \leq m \, ,$$

respectively.

Proof. For the proof of the statement about $S(\mathcal{S})$ we refer to [DR4]. Similarly, it was shown in [DR3] that $R(\mathcal{S})$ is quasi-hereditary. Thus we shall only prove here that $R(\mathcal{S})$ is replete and leave the verification of the statements about $M_r(\mathcal{S})$ and $M_\ell(\mathcal{S})$ to the reader.

To show that $R(\mathcal{S})$ is replete, we can combine Theorem 2 and the Proposition of Section 6 in [DR3]; then we get that $V(i)$ and $V^\circ(i)$ are both projective for $1 \leq i \leq n$; moreover, it follows from the construction that $e_iJe_jJe_k = 0$ for $i > j < k$; thus Propositions 2.4 and 2.4° imply that $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$ and similarly $V^\circ(i) \overset{t}{\subseteq} \operatorname{rad} P^\circ(i)$ for $1 \leq i \leq n$. $\square$

Let us note here that usually neither shallow, nor replete algebras are uniquely defined for a given species. In the following example we give the regular representations of algebras which are shallow, right medial, left medial and replete, but which are not isomorphic to the canonical algebras defined above.

Consider the ordered K-species given by the graph

$$\overset{1}{\bullet} \rightleftarrows \overset{2}{\bullet} \rightleftarrows \overset{3}{\bullet} \rightleftarrows \cdots \rightleftarrows \overset{n}{\bullet}$$

Then for $n = 4$ the following composition series charts describe the regular representations of shallow, right medial, left medial and replete algebras over the given species.

$$S_S = \begin{smallmatrix}1&\\&2\\1&\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&4\\3&\end{smallmatrix}\ ;$$

$$M_{r\,M_r} = \begin{smallmatrix}1&\\&2\\1&\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\\&2&\\1&&\end{smallmatrix} \oplus \begin{smallmatrix}&&3&\\&2&&4\\1&&3&\\&2&&\\1&&&\end{smallmatrix} \oplus \begin{smallmatrix}&&&4\\&&3&\\&2&&\\1&&&\end{smallmatrix}\ ;$$

$$M_{\ell\,M_\ell} = \begin{smallmatrix}1&&&\\&2&&\\1&&3&\\&2&&4\\&&3&\end{smallmatrix} \oplus \begin{smallmatrix}&2&&\\1&&3&\\&2&&4\\&&3&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&4\\3&\end{smallmatrix}\ ;$$

$$R_R = \begin{smallmatrix}1&&&\\&2&&\\1&&3&\\&2&&4\\1&&3&\\&2&&\\1&&&\end{smallmatrix} \oplus \begin{smallmatrix}&2&&\\1&&3&\\&2&&4\\1&&3&\\&2&&\\1&&&\end{smallmatrix} \oplus \begin{smallmatrix}&&3&\\&2&&4\\1&&3&\\&2&&\\1&&&\end{smallmatrix} \oplus \begin{smallmatrix}&&&4\\&&3&\\&2&&\\1&&&\end{smallmatrix}\ .$$

Finally, to justify the remarks preceding Theorem 3.4, we can state the following theorem, part of which can also be found in [DR4].

THEOREM 4.2. *Let A be a lean quasi-hereditary algebra with ordered species $\mathcal{S} = \mathcal{S}(A) = (D_1, D_2, \ldots, D_n\,;\, {}_iW_j\,,\ 1 \le i,j \le n)$. Then for $S = S(\mathcal{S})$ and $R = R(\mathcal{S})$ we have $\dim_K e_iSe_j \le \dim_K e_iAe_j \le \dim_K e_iRe_j$ for $1 \le i,j \le n$; in particular, $\dim_K S \le \dim_K A \le \dim_K R$ and equality holds if and only if A is either shallow or replete, respectively, with respect to the given order.*

Proof. For the inequalities concerning shallow algebras we refer to [DR4]. Actually, the minimality of the dimension of shallow algebras is valid even without the restriction that A is lean.

To show that $\dim_K e_iAe_j \le \dim_K e_iRe_j$ for $1 \le i,j \le n$, we shall proceed by downward induction on $i+j$. The case $i = j = n$ is trivial, since the quasi-heredity of A and R implies that $\dim_K e_nAe_n = \dim_K e_nRe_n = \dim_K D_n$. Assume now that $i + j < 2n$ and suppose that $i \le j$ (the other case will follow by symmetry). The fact that A and R are lean implies that $V_A(i) \simeq e_iA\varepsilon_{i+1}A \overset{t}{\subseteq} \operatorname{rad} P_A(i)$ and $V_R(i) \simeq e_iR\varepsilon_{i+1}R \overset{t}{\subseteq} \operatorname{rad} P_R(i)$. Thus for the projective cover of $V_A(i)$ we get $P\big(V_A(i)\big) \simeq \bigoplus_{k>i} P_A(k)^{d_{ik}}$ where $d_{ik} = \dim_{D_k} {}_iW_k\,$, and similarly, $V_R(i) \simeq \bigoplus_{k>i} P_R(k)^{d_{ik}}$ (since $V_R(i)$ is itself projective). Hence we have:

$$\dim_K V_A(i)e_j \le \dim_K P\big(V_A(i)\big)e_j = \sum_{k>i} d_{ik} \cdot \dim_K P_A(k)e_j\,,$$

and the induction hypothesis implies that the last term is not greater than $\sum_{k>i} d_{ik} \cdot \dim_K P_R(k)e_j = \dim_K V_R(i)e_j\,$. Since for $T = A$ or $T = R$ we have

$e_iTe_j = V_T(i)e_j$ for $i < j$ and $e_iTe_i \simeq V_T(i)e_i \oplus D_i$ as vector spaces, we have proved the inequality. Finally, if we have equality everywhere, then $V_A(i)$ (and $V_A^\circ(i)$) must be isomorphic to the projective cover $P\big(V_A(i)\big)$ (or $P\big(V_A^\circ(i)\big)$, respectively) for $1 \leq i \leq n$, thus implying that A is replete. □

References

[BG] Beilinson, A., Ginsburg, V., Mixed categories, Ext-duality and representations, preprint

[BGG] Berstein, J., Gelfand, I.M, Gelfand, S.I., A category of **g**-modules, Funct. Anal. Appl. **10** (1976), 87–92.

[BGS] Beilinson, A., Ginsburg, V., Soergel, W., Koszul duality patterns in representation theory, preprint.

[CPS1] Cline, E., Parshall, B.J., Scott, L.L., Finite dimensional algebras and highest weight categories, J. Reine Angew. Math. **391** (1988), 85–99.

[CPS2] Cline, E., Parshall, B.J., Scott, L.L., The homological dual of a highest weight category, preprint.

[D] Dyer, M.J., Kazhdan-Lusztig-Stanley polynomials and quadratic algebras I, preprint.

[DR1] Dlab, V., Ringel, C.M., Quasi-hereditary algebras, Illinois J. of Math. **35** (1989), 280–291.

[DR2] Dlab, V., Ringel, C.M., A construction for quasi-hereditary algebras, Compositio Math. **70** (1989), 155–175.

[DR3] Dlab, V., Ringel, C.M., Filtrations of right ideals related to projectivity of left ideals, Sém. d'Algèbre Dubreil–Malliavin, Springer LNM **1404** (1989), 95–107.

[DR4] Dlab, V., Ringel, C.M., The dimension of a quasi-hereditary algebra, in: *Topics in Algebra*, Vol **26**, Warsaw 1990, 263–271.

[DR5] Dlab, V., Ringel, C.M., The module theoretical approach to quasi-hereditary algebras, in: *Representations of Algebras and Related Topics* London Math. Soc. Lecture Note Series **168**, Cambridge Univ. Press 1992, 200–224.

[PS] Parshall, B.J., Scott, L.L., Derived categories, quasi-hereditary algebras, and algebraic groups, Proc. Ottawa–Moosonee Workshop, Carleton–Ottawa Math. Lecture Note Series **3** (1988), 1–105.

[S] Soergel, W., Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe, J. Amer. Math. Soc. **3** (1990), 421–445.

Mathematical Institute of the Hungarian Academy of Sciences, P.O.Box 127, 1364 Budapest, Hungary
E-mail address: h4134ago@ella.hu

Department of Mathematics and Statistics, Carleton University, Ottawa, Ontario, K1S 5B6, Canada
E-mail address: vdlab@ccs.carleton.ca

Department of Mathematics, Faculty of Transport Engineering, Technical University of Budapest, 1111 Budapest, Hungary
E-mail address: h4091luk@ella.hu