EXT-ALGEBRAS

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Abstract. The Ext-algebra A^* of a finite dimensional associative K-algebra A is studied with a motivation to establish conditions under which (i) the species of A and A^{*op} coincide and (ii) the quasi-heredity of A (or A^*) yields the quasi-heredity of A^* (or A, respectively). These questions are closely related to the Kazhdan–Lusztig Theory as presented by [CPS2].

1. Introduction

Throughout the paper A will denote a finite dimensional basic algebra over an arbitrary field K. Let us recall that the K*-species* $\mathcal{S}(A)$ of A is the system $(D_i : i \in I\,;\, {}_iW_j : i,j \in I)$ of finitely many division algebras D_i and D_i-D_j-bimodules ${}_iW_j$ so that $A/\operatorname{rad} A \simeq \prod_{i\in I} D_i$ and $\operatorname{rad} A/\operatorname{rad}^2 A \simeq \sum_{i,j\in I} {}_iW_j$. Thus, if $\{\, e_i \,|\, i \in I \,\}$ is a complete set of primitive orthogonal idempotents in A, and $\bar{e}_i$ denotes the image of e_i in $A/\operatorname{rad} A$, then $D_i = \bar{e}_i\big(A/radA\big)\bar{e}_i$ and ${}_iW_j = \bar{e}_i\big(\operatorname{rad} A/\operatorname{rad}^2 A\big)\bar{e}_j$. Notice that if $S(i)$ is the simple right A-module $e_iA/e_i\operatorname{rad} A$ then $D_i \simeq \operatorname{End}_A\big(S(i)\big)$ and ${}_iW_j \simeq \operatorname{Ext}^1_A\big(S^\circ(j), S^\circ(i)\big)$. If the field K is algebraically closed then one may speak about the *quiver* of the algebra A. For, all the division algebras are equal to K and the bimodules ${}_iW_j$ are just direct sums of copies of the regular bimodule K; hence, the complete information is contained in an oriented graph having I as its vertex set and $\dim_K {}_iW_j$ arrows from i to j.

Given an algebra A one may define the so-called *Ext-algebra* of A, denoted by A^*. This is a K-algebra whose underlying vector space is

$$\bigoplus_{k\geq 0}\ \bigoplus_{i,j\in I} \operatorname{Ext}^k_A\big(S(i), S(j)\big)\,,$$

with the multiplication defined via the Yoneda-product of exact sequences. Observe that A^* is finite dimensional if and only if $gl.dim\, A < \infty$; moreover the identity element of A^* is the sum of the primitive orthogonal idempotents

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$f_i = \mathrm{id}_{S(i)}$, $i \in I$. In analogy to $S(i)$ and $P(i) = e_iA$, denote by $S^{*\circ}(i)$ and $P^{*\circ}(i)$ the corresponding simple and indecomposable projective left A^*-modules.

Our principal objective is to study the connection between some of the properties of A and A^*, respectively. Some of our results are parallel to those of [CPS2] although our approach is somewhat different.

Most results presented here were reported by the authors on several occasions (Sherbrooke: May 1994, Prague: June 1994, Mexico City: August 1994). The proofs of the statements, together with some examples and further references to the graded situation will appear in a more detailed version elsewhere.

2. The species of Ext-algebras

First we will be dealing with the question of the species of A^* (more precisely, of A^{*op}). It is easy to see, that $\mathcal{S}(A) \subseteq \mathcal{S}(A^{*op})$. We will show that the fact that the species of these two algebras coincide is equivalent to some easy-to-describe property of the projective resolutions of the simple A-modules.

To this end we recall that a submodule X of Y is a *top submodule* (denoted by $X \overset{t}{\subseteq} Y$) if $\mathrm{rad}\, X = X \cap \mathrm{rad}\, Y$, i.e. the embedding of X into Y induces an embedding of $\mathrm{top}\, X$ into $\mathrm{top}\, Y$ (see [ADL1]). A filtration $X = X_1 \supseteq X_2 \supseteq \ldots \supseteq X_m$ of a module X is called a *top filtration* if $X_i \overset{t}{\subseteq} X$ for $1 \leq i \leq m$.

We shall also use the following notation. For an arbitrary module $X \in$ mod-A

$$\cdots \xrightarrow{d_{j+1}} \mathcal{P}_j(X) \xrightarrow{d_j} \cdots \xrightarrow{d_2} \mathcal{P}_1(X) \xrightarrow{d_1} \mathcal{P}_0(X) \xrightarrow{d_0} X \to 0$$

will denote a minimal projective resolution of X, with the corresponding syzygies $\Omega_{j+1}(X) = \mathrm{Ker}\, d_j$ for $j = 0, 1, \ldots$.

Now we may introduce the following subcategory of the category of finitely generated right A-modules mod-A.

Definition 2.1. We say that a module $X \in$ mod-A belongs to $\mathcal{C}^{(i)} = \mathcal{C}_A^{(i)}$ for some $i \in \mathbb{N}$ if $\Omega_j(X) \overset{t}{\subseteq} \mathrm{rad}\, \mathcal{P}_{j-1}(X)$ for $j = 1, 2, \ldots, i$. We may also define $\mathcal{C}^{(0)} =$ mod-A. The intersection of these subcategories will be denoted by $\mathcal{C}$; thus $\mathcal{C} = \mathcal{C}_A = \bigcap_{i=0}^{\infty} \mathcal{C}^{(i)}$. – Similarly, one may define the subcategory $\mathcal{C}_A^\circ \subset$ A-mod of left A-modules.

It is easy to see, that the definition does not depend on which particular minimal projective resolution of X was chosen.

The following proposition gives an important homological property of the elements of $\mathcal{C}^{(i)}$.

Proposition 2.2. *If $X \in \mathcal{C}^{(i)}$ then the natural maps $\mathrm{Ext}_A^k(\mathrm{top}\, X, S) \to \mathrm{Ext}_A^k(X, S)$ are surjective for every $0 \leq k \leq i$ and every simple module S.*

It turns out that with the addition of an easy necessary assumption, this property fully characterizes the elements of $\mathcal{C}^{(i)}$.

PROPOSITION 2.3. *Assume that every simple A-module S is in $\mathcal{C}_A$. Then a module X is an element of $\mathcal{C}_A^{(i)}$ if and only if the natural maps* $\mathrm{Ext}_A^k(\mathrm{top}\, X, S) \to \mathrm{Ext}_A^k(X, S)$ *are surjective for every $0 \leq k \leq i$ and S simple module.*

Proposition 2.2 leads to a full answer as to when the species of A and A^{*op} coincide.

THEOREM 2.4. *The following are equivalent for an algebra A.*
(a) $S \in \mathcal{C}_A$ for every simple right module S;
(b) $S^\circ \in \mathcal{C}_A^\circ$ for every simple left module S°;
(c) $\mathcal{S}(A) = \mathcal{S}(A^{*op})$.

3. The functor $\mathrm{Ext}^* : \text{mod-}A \to A^*\text{-mod}$

We shall assume in this section that the Ext-algebra A^* of the finite dimensional algebra A is itself finite dimensional, i. e. $gl.dim\, A < \infty$.

Let $\hat{S}$ denote the direct sum of all simple right A-modules, i. e. $\hat{S} = \bigoplus_{i \in I} S(i)$. Then we may define a contravariant functor $\mathrm{Ext}^* : \text{mod-}A \to A^*\text{-mod}$ by taking the direct sum of the functors $\mathrm{Ext}^k(-, \hat{S})$ for $k \geq 0$. Actually, the modules $\mathrm{Ext}^*(X)$ will have a natural grading, with the morphisms $\mathrm{Ext}^*(f)$ preserving this grading, hence we have a functor into $A^*\text{-mod}_{gr}$. For a module $X \in A^*\text{-mod}_{gr}$, let $X[j]$ denote the shifted graded module, i. e. $X[j]_i = X_{i-j}$. We have the following exactness properties of Ext^*.

LEMMA 3.1. *Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in* mod-A.
(a) Assume $X \overset{t}{\subseteq} Y$. If $X \in \mathcal{C}_A$ then the sequence $0 \to \mathrm{Ext}^*(Z) \to \mathrm{Ext}^*(Y) \to \mathrm{Ext}^*(X) \to 0$ is exact; if in addition $Z \in \mathcal{C}_A$, then $\mathrm{Ext}^*(Z) \overset{t}{\subseteq} \mathrm{Ext}^*(Y)$.
(b) Assume $X \subseteq \mathrm{rad}\, Y$. If $Y \in \mathcal{C}_A$ then the sequence $0 \to \mathrm{Ext}^*(X)[1] \to \mathrm{Ext}^*(Z) \to \mathrm{Ext}^*(Y) \to 0$ is exact; if in addition $Z \in \mathcal{C}_A$, then $\mathrm{Ext}^*(X)[1] \subseteq \mathrm{rad}\, \mathrm{Ext}^*(Z)$.

Based on this lemma, we get the following propositions.

PROPOSITION 3.2. *If $X, \mathrm{rad}\, X \in \mathcal{C}_A$ then $\mathrm{Ext}^*(X) \in \mathcal{C}_{A^*}^{(1)^\circ}$. Thus if $\mathrm{rad}^i X \in \mathcal{C}_A$ for every i then $\mathrm{Ext}^*(X) \in \mathcal{C}_{A^*}^\circ$.*

PROPOSITION 3.3. *(a)* $\mathrm{Ext}^*\big(S(i)\big) = P^{*\circ}(i)$.
(b) $\mathrm{Ext}^*\big(P(i)\big) = S^{*\circ}(i)$.
(c) $\mathrm{Ext}^*\big(\mathrm{rad}\, P(i)\big)[1] = \mathrm{rad}\, P^{*\circ}(i)$.

4. Ext-algebras and quasi-heredity

To speak about the quasi-heredity of an algebra A, one must impose a (partial) order on the set $\{ S(i) \mid i \in I \}$ of simple right A-modules (or equivalently, on the given complete set of primitive orthogonal idempotents). Actually, without loss of generality we may assume that we have a total order on the index set I. Thus assume that $I = \{ 1, 2, \dots, n \}$ with the natural order. We shall write $\mathbf{e} = (e_1, e_2, \dots, e_n)$ for the corresponding ordered set of primitive orthogonal idempotents and we define $\varepsilon_i = e_i + e_{i+1} + \dots + e_n$, $\varepsilon_{n+1} = 0$. Recall that $P(i)$ denotes the projective cover of the simple module $S(i)$. Consider the *trace filtration* of A:

$$A = A\varepsilon_1 A \supseteq A\varepsilon_2 A \supseteq \dots \supseteq A\varepsilon_n A \supseteq 0\,.$$

We say that A is *quasi-hereditary* with respect to I (or briefly, $(A, \mathbf{e})$ is quasi-hereditary) if each of the so called *standard right modules* $e_iA/e_iA\varepsilon_{i+1}A$, denoted by $\Delta(i)$ is *Schurian* (i. e. it has a semisimple endomorphism ring) and the quotients of the trace filtration $A\varepsilon_iA/A\varepsilon_{i+1}A$ as right modules are direct sums of the corresponding standard modules. In addition, we say that A is *lean* with respect to this order if $\Delta(i) \in \mathcal{C}_A^{(1)}$ and $\Delta^\circ(i) \in \mathcal{C}_A^{(1)^\circ}$ for all $i \in I$. (Here $\Delta^\circ(i)$ stands for the corresponding standard left module.) We consider the following canonical exact sequences:

$$0 \to V(i) \to P(i) \to \Delta(i) \to 0 \qquad \text{and} \qquad 0 \to U(i) \to \Delta(i) \to S(i) \to 0\,.$$

For the basic properties of quasi-hereditary algebras, we refer to [CPS1], [DR1], [DR2] or [DK] and of lean algebras to [ADL1], [ADL2]. Canonical constructions for the so-called *shallow*, *replete* and *medial algebras* are also described there.

We have already noticed that the simple types of (right) A-modules are in one-to-one correspondence with the simple types of (left) A^*-modules; the corresponding idempotent to the primitive idempotent $e_i \in A$ is the element $f_i = \mathrm{id}_{S(i)} \in A^*$. Having fixed the order $\mathbf{e} = (e_1, e_2, \dots, e_n)$ for A we shall consider the reverse order $\mathbf{f} = (f_n, f_{n-1}, \dots, f_1)$ for A^*; write $\varphi = f_i + f_{i-1} + \dots + f_1$ and $\varphi_0 = 0$.

One of the key observations in recognizing the quasi-heredity of A^* is the following lemma.

Lemma 4.1. *Assume that $(A, \mathbf{e})$ is quasi-hereditary with $\Delta(i) \in \mathcal{C}_A$ and $U(i) \in \mathcal{C}_A$ for $1 \le i \le n$. Then the left standard module $\Delta^{*\circ}(i)$ of $(A^*, \mathbf{f})$ is Schurian and $\Delta^{*\circ}(i) \simeq \mathrm{Ext}^*\big(\Delta(i)\big)$. Furthermore, with similar notation, $\mathrm{Ext}^*\big(U(i)\big)[1] \simeq V^{*\circ}(i)$ and $\mathrm{Ext}^*\big(V(i)\big)[1] \simeq U^{*\circ}(i)$.*

We can now state the following sufficient condition for a quasi-hereditary algebra to have a quasi-hereditary Ext-algebra.

Definition 4.2. An algebra $(A, \mathbf{e})$ is said to be *solid*, if the following conditions are satisfied:

(1) $\Delta(i)$ is Schurian;
(2) $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$;
(3) $U(i)$ has a top filtration by $S(j)$'s and $\Delta(j)$'s for $j < i$;
(4) $V(i)$ has a top filtration by $\Delta(j)$'s and $P(j)$'s for $j > i$.

Lemma 4.3. *If $(A, \mathbf{e})$ is solid then it is a lean quasi-hereditary algebra with $S(i), \Delta(i), U(i) \in \mathcal{C}_A$ for $1 \leq i \leq n$.*

Theorem 4.4. *Let $(A, \mathbf{e})$ be a solid algebra. Then:*

(a) $(A^{*op}, \mathbf{f})$ is a solid algebra (hence quasi-hereditary), and
(b) $\mathcal{S}(A) = \mathcal{S}(A^{*op})$, $\dim_K A^{**} = \dim_K A$, $(\varepsilon_i A \varepsilon_i)^* \simeq A^*/(A^* \varphi_{i-1} A^*)$ and $\big(A/(A\varepsilon_i A)\big)^* \simeq \varphi_{i-1} A^* \varphi_{i-1}$.

Corollary 4.5. *If the algebra $(A, \mathbf{e})$ is shallow (left medial, right medial or replete) then $(A^{*op}, \mathbf{f})$ is replete (left medial, right medial or shallow, respectively) on the same species.*

5. Ext-algebras of monomial algebras

We can get a more complete picture of the situation in the case of monomial algebras. Here the principal tool in the understanding is the existence of a multiplicative basis for A^*, consisting of some paths in the quiver of A (see [GZ]). Thus we shall assume now that A is *monomial*, i. e. $A = K\Gamma/R$, where Γ is a quiver with R the set of relations which is generated by some paths of length at least 2. First, we have an extension of Theorem 2.4 about the quiver of A^{*op}.

Theorem 5.1. *Let $A \simeq K\Gamma/R$ be a monomial algebra. Then the following are equivalent:*

(a) $S(i) \in \mathcal{C}_A$ for $1 \leq i \leq n$;
(b) A and A^{*op} have the same quiver;
(c) A is quadratic (i. e. the set of relations R is generated by paths of length 2);
(d) $\operatorname{Ext}^2_A(\hat{S}, \hat{S}) \subseteq \operatorname{rad}^2(A^*)$.

If $(A, \mathbf{e})$ is in addition lean with Schurian standard modules, then conditions (a)–(d) are all equivalent to:

(e) $\Delta(i) \in \mathcal{C}$, $\Delta^\circ(i) \in \mathcal{C}^\circ$ for $1 \leq i \leq n$.

On the question of quasi-heredity we have the following results.

Theorem 5.2. *Let $A = K\Gamma/R$ be a monomial algebra with gl.dim $A < \infty$. Then $(A^*, \mathbf{f})$ is quasi-hereditary if and only if $(A, \mathbf{e})$ is lean with Schurian standard modules.*

THEOREM 5.3. *Let $A = K\Gamma/R$ be a monomial algebra. If $(A, \mathbf{e})$ is quasi-hereditary then either $(A^*, \mathbf{f})$ is lean with Schurian standard modules or the quiver of A^* has a loop.*

Thus from the previous two theorems we get the following corollary.

COROLLARY 5.4. *Let $A = K\Gamma/R$ be a monomial algebra. Then if $(A, \mathbf{e})$ is lean and quasi-hereditary, then so is $(A^*, \mathbf{f})$.*

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