ENTRANCE EXAMINATION – PHD

Duration: 360 minutes

During your work:

(1) you MAY use textbooks or course notes;

(2) you SHOULD NOT use external help;

(3) you SHOULD NOT use electronic devices for symbolic computations;

(4) you SHOULD NOT use the internet for search or communication.

BY WRITING THIS TEST,
YOU AGREE TO THE TERMS LISTED ABOVE.

This examination contains 15 pages, including this top page and a DATA PAGE.

Pages 3-15 contain tests on the following topics:

0. (BMA) BASIC MATHEMATICS (2 pages)
1. (ALG) ALGEBRA
2. (GEO) GEOMETRY
3. (COM) COMBINATORICS
4. (ANA) ANALYSIS (REAL AND COMPLEX)
5. (PRO) PROBABILITY THEORY
6. (OPR) OPERATIONS RESEARCH
7. (SET) SET THEORY
8. (MFA) MEASURE THEORY / FUNCTIONAL ANALYSIS
9. (DEQ) DIFFERENTIAL EQUATIONS
10. (NUM) NUMERICAL ANALYSIS
11. (TDG) TOPOLOGY / DIFFERENTIAL GEOMETRY

- **You have to do the test on BASIC MATHEMATICS.**
- **You also have to choose 3 further topics ACCORDING TO THE RULES ON THE NEXT PAGE (DATA PAGE) from the subjects 1–11 and attempt all problems from the chosen topics. Each topic is worth 100 marks.**

- **Please, make sure that your presentation is clear and readable. Send us your work even if you have not solved each problem or have only partial results in a certain problem. Unless otherwise stated, you SHOULD JUSTIFY your answers.**
- **Please, write on top of each page your NAME, COUNTRY and also indicate the number of the PROBLEM for which you provide a solution on the given page (e.g. PHD/ANALYSIS/2 for the second problem in analysis).**
- **Send the scanned copy of your solutions (together with the DATA PAGE – see page 2) WITHIN 6 HOURS of receiving the problem sheet to:**

`agoston@cs.elte.hu`

A confirmation letter will be sent within the next 24 hours.

GOOD LUCK!

DATA PAGE – PHD

Please, fill the table and send it back together with your solutions.
(If you do not have a printer, copy the data from the tables by hand.)

FULL NAME	
HOME UNIVERSITY	
CITY	
COUNTRY	

Please, choose an area of your future studies (in agreement with your motivation letter) and choose the test subjects accordingly. Please, mark your choice by X in the appropriate box.

AREA OF RESEARCH	BASIC	MANDATORY	ELECTIVE	FREE (give subject name)
☐ A. ALGEBRA / NUMBER THEORY	0. BMA	1. ALG	☐ 3. COM or ☐ 4. ANA	
☐ B. GEOMETRY / TOPOLOGY	0. BMA	2. GEO	☐ 8. MFA or ☐ 11. TDG	
☐ C. DISCRETE MATHEMATICS	0. BMA	3. COM	☐ 6. OPR or ☐ 7. SET	
☐ D. ANALYSIS	0. BMA	4. ANA	☐ 8. MFA or ☐ 9. DEQ	
☐ E. APPLIED ANALYSIS	0. BMA	4. ANA	☐ 9. DEQ or ☐ 10. NUM	
☐ F. PROBABILITY AND STATISTICS	0. BMA	5. PRO	☐ 4. ANA or ☐ 8. MFA	
☐ G. OPERATIONS RESEARCH	0. BMA	6. OPR	☐ 3. COM or ☐ 4. ANA	
☐ H. SET THEORY / MATH. LOGIC	0. BMA	7. SET	☐ 1. ALG or ☐ 4. ANA	

SOLUTIONS to the test from BASIC MATHEMATICS

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
☐	☐	☐	☐	☐	☐	☐	☐	☐	☐

0. (BMA) BASIC MATHEMATICS: 10 multiple choice questions

Each problem has exactly ONE CORRECT ANSWER.
Please, send us your solutions by filling the boxes on the DATA PAGE.
You do not have to justify your answers in this part.

0/1. Which of the following statements is **equivalent** to the statement "A implies B" ?

a) If A is false then B is false.
b) If B is false then A is false.
c) A is true and B is false.
d) B implies A.

0/2. Which of the following statements is **true**?

a) $2.99999... < 3$
b) $2.99999... = 3$
c) $2.99999... < 3.00000...$
d) $3 < 3.00000...$

0/3. Which of the following statements is **false**?

a) The sum of two rational numbers is always rational.
b) The sum of a rational and an irrational number is always irrational.
c) The difference of a rational and an irrational number is always irrational.
d) The sum of two irrational numbers is always irrational.

0/4. Which ending makes the following statement **true**:

"$x > y$ implies $x^2 > 25$ if and only if..."

a) $y > 5$
b) $y \geq 5$
c) $y < 5$
d) $y \leq 5$

0/5. Which of the following statements is **true**?

a) Every nonempty set of real numbers has an upper bound.
b) A set of real numbers can have at most one upper bound.
c) Every bounded set of real numbers has exactly one upper bound.
d) Every bounded set of real numbers has exactly one least upper bound.

THIS TEST CONTINUES ON THE NEXT PAGE!

0/6. Let A and B be nonempty sets and let $f(x, y)$ be an integer valued function on $A \times B$. What is the logical connection between the following two statements?

(P) For every $a \in A$ there exists an element $b \in B$ such that $f(a, b) > 0$.
(Q) There exists an element $b \in B$ such that for every $a \in A$ we have $f(a, b) > 0$.

a) (P) implies (Q).
b) (Q) implies (P).
c) (P) and (Q) are equivalent.
d) None of the statements implies the other.

0/7. Which of the following statements is **false**?

a) The minimum of a set is always an element of the set.
b) The minimum of a set is always a lower bound of the set.
c) The infimum of a set is always an element of the set.
d) The infimum of a set is always a lower bound of the set.

0/8. Which of the following is **impossible** if $a_n \to \infty$ and $b_n \to 0$?

a) $a_n/b_n \to \infty$.
b) $a_n/b_n \to -\infty$.
c) a_n/b_n has a finite limit.
d) a_n/b_n has no finite or infinite limit.

0/9. Which if the following statements is **true**?

a) Every increasing infinite sequence of real numbers has a convergent subsequence.
b) If an infinite sequence of real numbers has no convergent subsequence then the sequence cannot be bounded.
c) Every infinite sequence of real numbers has a convergent subsequence.
d) Every subsequence of a divergent sequence of real numbers is divergent.

0/10. How many subsets A of $\{1, 2, \ldots, 10\}$ have the following property:

$$1 \in A \implies 2 \in A$$

a) 256;
b) 512;
c) 768;
d) 1024.

1. (ALG) ALGEBRA: 4 problems

1/1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis of the real vector space V. Take the linear transformation $\varphi : V \to V$ defined by $\varphi(\mathbf{b}_1) = \varphi(\mathbf{b}_2) = \varphi(\mathbf{b}_3) = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$.

a) Write the matrix $A = [\varphi]_{\mathcal{B}}$ of the transformation φ relative to the basis $\mathcal{B}$.

b) Find the characteristic polynomial, eigenvalues and eigenvectors of φ.

c) Find an orthonormal basis for every eigenspace of φ.

d) Is there an orthogonal matrix $S \in \mathbb{R}^{3\times 3}$ such that $S^{-1}AS$ is diagonal? (Note that a matrix $B \in \mathbb{R}^{n\times n}$ is orthogonal if the transpose of B is equal to the inverse of B.)

1/2. If $\varphi : V \to V$ is a linear transformation of the vector space V then a subspace $U \leq V$ is called φ-*invariant* if for every vector $\mathbf{u} \in U$ we have $\varphi(\mathbf{u}) \in U$. For example, $\{\mathbf{0}\}$ and V are always φ-invariant. (These are called the *trivial* invariant subspaces.)

a) Give an example of a vector space V (with $\dim V > 1$) and a linear transformation φ so that there are no non-trivial φ-invariant subspaces in V.

b) Let V be a vector space of uncountable dimension and let $\varphi : V \to V$ be a linear transformation. Show that V must have a non-trivial φ-invariant subspace.
(Justify your answer in both parts.)

1/3. Let S_n be the group of all permutations on n letters and let s_n denote the number of subgroups in S_n. Prove that for $n \geq 3$ we have $(n-2)! < s_n < 2^{n!-1}$.

1/4. A (not necessarily commutative) ring R (possibly without an identity element) is called (von Neumann) *regular* if for every $a \in R$ there exists an element $x \in R$ such that $axa = a$. Prove that for an arbitrary ring R and an ideal I of R:

R is regular $\iff$ both the ideal I as a ring and the quotient ring R/I are regular.

2. (GEO) GEOMETRY: 4 problems

2/1. Let $ABCD$ be a square in the plane Σ and consider a pyramid over $ABCD$ with fifth vertex O, for which the orthogonal projection of O onto Σ is the center of the square $ABCD$, and the edges OA, OB, OC, and OD intersect the plane Σ at angle $60°$.

Assume that the lateral edges of the pyramid are longer than 4 and choose the points $A' \in [O, A]$, $B' \in [O, B]$, $C' \in [O, C]$, $D' \in [O, D]$ so that the segments $[A, A']$, $[B, B']$, $[C, C']$, $[D, D']$ have length 4.

We know that the diagonal $[A, C']$ of the truncated pyramid $ABCDA'B'C'D'$ has length $2\sqrt{19}$. Compute the lengths of the edges $[A, B]$ and $[A', B']$.

2/2. Let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ be non-zero vectors in the Euclidean space, $\alpha_{i,j}$ be the angle enclosed by $\mathbf{a}_i$ and $\mathbf{a}_j$. Show that

$$\cos(\alpha_{1,2}) + \cos(\alpha_{2,3}) + \cos(\alpha_{3,1}) \geq -\frac{3}{2}.$$

2/3. Assume that a Cartesian coordinate system is fixed in the 3-dimensional Euclidean space. Let S be an arbitrary plane through the origin, and $\Pi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map, which assigns to a point $(x, y, z) \in \mathbb{R}^3$ its orthogonal projection onto S. Show that the trace of Π is 2.

2/4. If $A \subset \mathbb{R}^n$, then let $\Phi(A)$ be the union of all closed segments connecting two points of A, i.e.,

$$\Phi(A) = \bigcup_{\mathbf{p},\mathbf{q} \in A} [\mathbf{p}, \mathbf{q}],$$

and define the sequence of sets $A_0, A_1, \ldots$ recursively by $A_0 = A$, and $A_{k+1} = \Phi(A_k)$.

Show that for any $A \subset \mathbb{R}^n$, A_n is the convex hull of A.

3. (COM) COMBINATORICS: 5 problems

3/1. Show that for any non-negative integer n, the equation

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^n = n!$$

holds.

3/2. At an international meeting, each of some 11 countries are represented by 5 diplomats who will be sitting around a round table. Is it possible to seat the diplomats so that for any two countries some of their diplomats are sitting next to each other?

3/3. Show that any positive integer n has a multiple which (written in decimal system) contains only digits 0 and 1.

3/4. Show that a 3-regular simple graph is k-connected (that is, k-vertex-connected) if and only if it is k-edge-connected.

3/5. Let G be a simple graph on n vertices which does not contain a cycle of length three or four.

a) Show that G has at most $\frac{n}{2}\sqrt{n-1}$ edges and characterize examples that attain equality in the bound.

b) Show two different examples of graphs meeting the bound.

4. (ANA) ANALYSIS: 5 problems

4/1. Compute the distance between the point $(5,5)$ and the hyperbola $xy=4$.

4/2. For any $f:\mathbb{R}\to\mathbb{R}$ and for any $n\in\mathbb{N}$ one defines $f^{\circ n}$ recursively by $f^{\circ 1}:=f$ and $f^{\circ n+1}:=f\circ f^{\circ n}$.

Does there exist some $c\in\mathbb{R}$ such that the polynomial $p(x)=x^{2018}+cx^2$ has the property that for any $a\in\mathbb{R}$ the sequence $a_n:=p^{\circ n}(a)$ either diverges to infinity or converges to 0?

4/3. Is the function

$$f(x,y,z)=\begin{cases}\dfrac{xyz}{x^2+y^2+z^2} & \text{if } (x,y,z)\neq(0,0,0)\\ 0 & \text{if } (x,y,z)=(0,0,0)\end{cases}$$

continuous? Is f differentiable at $(0,0,0)$?

4/4. Calculate

$$\int_0^1\int_y^1 e^{x^2}\,\mathrm{d}x\mathrm{d}y.$$

4/5. Let C be the unit circle in the complex plane with positive orientation. Calculate

$$\int_C \frac{\mathrm{d}z}{\sin^3(z)}.$$

5. (PRO) PROBABILITY THEORY: 4 problems

5/1. We roll a fair die until we get 6 for the second time. Let X_1 be the number of rolls until we get 6 for the first time, and let X_2 be the number of rolls until we get 6 for the second time (e.g. in case of the sequence 364126 we have $X_1 = 2$ and $X_2 = 6$). Find the conditional expectation of X_1 given that $X_2 = 5$.

5/2. Let X be a random variable with the following density function:

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1; \\ \dfrac{3}{14}x^2 & \text{if } 1 \le x < 2; \\ 0 & \text{otherwise.} \end{cases}$$

Find the variance of X.

5/3. Suppose that a transport company operates 40 trams and 80 buses. Every day, independently of each other, each tram breaks down with probability 0.01, and each bus breaks down with probability 0.02 (independently of each other and of the trams). Let X be the number of trams breaking down during a day, and Y be the number of buses breaking down on the same day. Calculate the covariance of X and $X \cdot Y$.

5/4. Let $X_1, X_2, \ldots$ be a sequence of independent random variables distributed uniformly on the interval $[0, 2]$. Find all possible values of the following limit, where α is a positive number:

$$\lim_{n\to\infty} \mathbb{P}\left(\left| \sum_{i=1}^{n^2} X_i - n^2 \right| \le n^\alpha \right).$$

6. (OPR) OPERATIONS RESEARCH: 5 problems

6/1. Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c_1 \in \mathbb{R}^n,\ c_2 \in \mathbb{R}^n, \ldots,\ c_k \in \mathbb{R}^n$. Write the following nonlinear problem as a linear program:

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{k} |c_i^T x| \\ \text{subject to} \quad & Ax = b \\ & x \geq \mathbf{0}. \end{aligned}$$

6/2. Let $G = (V, E)$ be an undirected graph, and let c_e ($e \in E$) be edge costs. The cost of an edge set $F \subseteq E$ is $\sum_{e\in F} c_e$. An edge set $F \subseteq E$ is a *2-matching* if every node is incident to at most 2 edges in F. Formulate the minimum cost 2-matching problem as an integer program (IP). Write the dual linear program of its LP-relaxation.

6/3. Let $G = (S, T; E)$ be a bipartite graph. For $v \in S \cup T$, let $d(v)$ denote the degree of v in G. Prove that, for any positive integer k, the edges of G can be colored by k colors such that the number of edges of each color incident to a node v is $\lfloor d(v)/k \rfloor$ or $\lceil d(v)/k \rceil$.

6/4. Solve the following linear program using the simplex method.

$$\begin{aligned} \text{Maximize} \quad & 3x_2 + 4x_3 \\ \text{subject to} \quad & x_1 + 3x_2 + 4x_3 \leq 10 \\ & x_1 + x_2 + 3x_3 = 8 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

6/5. Let $T = (V, E)$ be a tree. For every node $v \in V$, we are given a finite list L_v of possible colors. Give a polynomial-time algorithm that decides if it is possible to choose one color for each node v from the list L_v, so that the resulting coloring of the tree is proper, i.e. no adjacent nodes receive the same color.

7. (SET) SET THEORY: 4 problems

7/1. Let $(A, <)$ be an ordered set in which for each element $x \in A$ either the set $\{y \in A \mid y < x\}$ or the set $\{y \in A \mid y > x\}$ is finite. Show that A is countable.

7/2. Show that there are exactly continuum many planar isosceles triangles.

7/3. Let A_n $(n = 0, 1, 2, \ldots)$ be infinite sets. Show that there is a set B such that $B \cap A_n \neq \emptyset$ and $A_n \not\subseteq B$ for each n.

7/4. If A is a set, what is $\bigcup \mathcal{P}(A)$? ($\mathcal{P}(A)$ is the power set of A.)

8. (MFA) MEASURE THEORY / FUNCTIONAL ANALYSIS: 4 problems

8/1. a) Let $A \subset [0,1]$ be a Lebesgue-measurable set and $\varepsilon > 0$. Prove that there exists a set $I \subset [0,1]$ such that I is the union of finitely many intervals with rational end-points, $\lambda(A \setminus I) < \varepsilon$ and $\lambda(I \setminus A) < \varepsilon$.

b) Can one find an uncountable family $\mathcal{F}$ of Lebesgue measurable sets in $[0,1]$ such that every set $A \in \mathcal{F}$ satisfies $\lambda(A) = 1/2$, and every two different sets $A, B \in \mathcal{F}$ satisfy $\lambda(A \cap B) = 1/4$?

8/2. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose that $f : X \to [0, \infty]$ is a measurable function with $\int_X f \, \mathrm{d}\mu = 1$. For all real $\alpha > 0$, prove that the limit

$$\lim_{h \to 0+} \int_X \frac{\log\left(1 + (h \cdot f)^\alpha\right)}{h} \, \mathrm{d}\mu$$

exists and compute it.

8/3. Let $\mathcal{H}$ be a Hilbert space and $A \in B(\mathcal{H})$. Show that if

$$\sup_{x \in \mathcal{H}, \|x\|=1} |(Ax, x)| = \|A\|,$$

then $r(A) = \|A\|$ holds for the spectral radius $r(A)$ of A.

8/4. Let X be a normed space and $f \in X^*$ a non-zero bounded functional. Show that

$$\|f\| = \sup \left\{ \frac{1}{\|x\|} : x \in X, f(x) = 1 \right\}.$$

9. (DEQ) DIFFERENTIAL EQUATIONS: 4 problems

9/1. Sketch the global phase portrait of the two dimensional autonomous system

$$\begin{cases} \dot{x} = x^2 + y^2 - 2, \\ \dot{y} = y + x^2, \end{cases}$$

and determine the stability of the steady states.

9/2. Find the solution of the second-order differential equation

$$\ddot{x}(t) + 2\dot{x}(t) - 3x(t) = t^2 e^t$$

which satisfies the initial conditions $x(0) = 0$, $\dot{x}(0) = 1$.

9/3. Sketch the bifurcation diagram and phase lines for the ordinary differential equation

$$\dot{x}(t) = 2\lambda x(t) + 3x^2(t),$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter, and classify the bifurcation that occurs.

9/4. Find the solution of the inhomogeneous heat equation

$$\partial_t u(x,t) - \partial_x^2 u(x,t) = t \sin 2x$$

in the domain $(0,\pi) \times (0,\infty)$ which satisfies the initial condition $u(x,0) = \sin x$ for $x \in [0,\pi]$ and the boundary condition $u(0,t) = u(\pi,t) = 0$ for $t > 0$.

10. (NUM) NUMERICAL ANALYSIS: 4 problems

10/1. Let us consider the system of equations $Ax = b$, where

$$A = \begin{pmatrix} 5 & 0 & -2 \\ -4 & 8 & 2 \\ 0 & 5 & 9 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -1 \\ 18 \\ 37 \end{pmatrix}$$

a) Are the Jacobi and Gauss-Seidel iterative methods convergent for the above problem?
b) Let us apply these iterative methods from the starting vector $x = (0, 0, 0)^T$. Write the first two iterates.

10/2. Let us consider the function

$$f(x) = \frac{x}{2} + \frac{1}{x}, \; x \in [1, 2].$$

a) Can we apply the Banach fixed-point theorem to solve the equation $x = f(x)$?
b) If yes, then start the fixed-point iteration from $x_0 = 1$. At least how many iteration steps do we need so that the numerical solution approaches the true solution within the error 10^{-5}?

10/3. Let us consider the Runge–Kutta method

$$y_{n+1} = y_n + h\left[\frac{1}{4}f(t_n, y_n) + \frac{3}{4}f\left(t_n + \frac{2}{3}h, y_n + \frac{2}{3}f\Big(t_n + \frac{1}{3}h, y_n + \frac{1}{3}f(t_n, y_n)\Big)\right)\right].$$

a) Apply the above method to the problem

$$\begin{cases} y'(t) = 1 - y(t), \; t \in [0, 1] \\ y(0) = 0 \end{cases}$$

and calculate its numerical solution at the time $t = 1$ with $h = 1/2$.
b) Determine the stability function of the method in the sense of Dahlquist's test problem. Is it an A-stable method?

10/4. Consider the boundary value problem

$$\begin{cases} u''(x) - u(x) = e^x, \; x \in (0, 1) \\ u'(0) = 1 \\ u(1) = 1 \end{cases}$$

By using the standard second order central difference scheme for the second derivative and the right difference quotient for the first derivative write the corresponding linear algebraic system if we use a uniform grid with $N + 2$ points.

11. (TDG) TOPOLOGY / DIFFERENTIAL GEOMETRY: 5 problems

11/1. Let X be a topological space, $A \subset X$ be a closed subspace, which is locally connected with respect to the subspace topology, and let $A_i \subseteq A$, $i \in I$, be the connected components of A. Show that for any $J \subseteq I$, the union $\bigcup_{i \in J} A_i$ is closed in X.

(A topological space A is said to be *locally connected* if for any point $p \in A$, every neighbourhood of p (in A) contains a connected neighbourhood of p.)

11/2. Let X be the union of 4 balls pairwise touching one another, $x_0 \in X$ be an arbitrary point. Compute the fundamental group $\pi_1(X, x_0)$.

11/3. Let $\gamma\colon [a, b] \to \mathbb{R}^3$ be a smooth parameterized curve for which $\gamma''(t)$ is parallel to $\gamma(t)$ for all $t \in [a, b]$. Show that the cross product $\gamma \times \gamma'$ is constant.

11/4. Construct a nowhere zero tangential vector field on the 3-dimensional sphere

$$\mathbb{S}^3 = \{\mathbf{x} \in \mathbb{R}^4 \mid \|\mathbf{x}\| = 1\}.$$

11/5. Does the equation $x^2 = y^2$ define a 1-dimensional manifold in the plane? Justify your answer.