ENTRANCE EXAMINATION – MSC

Duration: 360 minutes

During your work:

(1) you MAY use textbooks or course notes;
(2) you SHOULD NOT use external help;
(3) you SHOULD NOT use electronic devices for symbolic computations;
(4) you SHOULD NOT use the internet for search or communication.

BY WRITING THIS TEST,
YOU AGREE TO THE TERMS LISTED ABOVE.

This examination contains 9 pages, including this top page and a DATA PAGE.

Pages 3-9 contain tests on the following topics:

0. **BASIC MATHEMATICS (2 pages)**
1. **ALGEBRA**
2. **GEOMETRY**
3. **COMBINATORICS**
4. **ANALYSIS**
5. **PROBABILITY THEORY**

- **You have to do the test on BASIC MATHEMATICS.**
- **You also have to choose 3 further topics from the subjects 1–5 and attempt all problems from the chosen topics. Each topic is worth 100 marks.**

- **Please, make sure that your presentation is clear and readable. Send us your work even if you have not solved each problem or have only partial results in a certain problem. Unless otherwise stated, you SHOULD JUSTIFY your answers.**
- **Please, write on top of each page your NAME, COUNTRY and also indicate the number of the PROBLEM for which you provide a solution on the given page (e.g. MSC/ANALYSIS/2 for the second problem in analysis).**
- **Send the scanned copy of your solutions (together with the DATA PAGE – see page 2) WITHIN 6 HOURS of receiving the problem sheet to:**

`agoston@cs.elte.hu`

A confirmation letter will be sent within the next 24 hours.

GOOD LUCK!

DATA PAGE – MSC

Please, fill the table and send it back together with your solutions
(If you do not have a printer, copy the data from the tables by hand)

FULL NAME	
HOME UNIVERSITY	
CITY	
COUNTRY	

Mark by X problems for which you are sending a solution.

		1.	2.	3.	4.	5.
☐	**1. ALGEBRA**	☐	☐	☐	☐	
☐	**2. GEOMETRY**	☐	☐	☐	☐	
☐	**3. COMBINATORICS**	☐	☐	☐	☐	☐
☐	**4. ANALYSIS**	☐	☐	☐	☐	☐
☐	**5. PROBABILITY THEORY**	☐	☐	☐	☐	

SOLUTIONS to the test from BASIC MATHEMATICS

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
☐	☐	☐	☐	☐	☐	☐	☐	☐	☐

0. BASIC MATHEMATICS: 10 multiple choice questions

Each problem has exactly ONE CORRECT ANSWER.
Please, send us your solutions by filling the boxes on the DATA PAGE.
You do not have to justify your answers in this part.

0/1. Which of the following statements is **equivalent** to the statement "A implies B" ?
- a) If A is false then B is false.
- b) If B is false then A is false.
- c) A is true and B is false.
- d) B implies A.

0/2. Which of the following statements is **true**?
- a) $2.99999... < 3$
- b) $2.99999... = 3$
- c) $2.99999... < 3.00000...$
- d) $3 < 3.00000...$

0/3. Which of the following statements is **false**?
- a) The sum of two rational numbers is always rational.
- b) The sum of a rational and an irrational number is always irrational.
- c) The difference of a rational and an irrational number is always irrational.
- d) The sum of two irrational numbers is always irrational.

0/4. Which ending makes the following statement true:

"$x > y$ implies $x^2 > 25$ if and only if…"
- a) $y > 5$
- b) $y \geq 5$
- c) $y < 5$
- d) $y \leq 5$

0/5. Which of the following statements is **true**?
- a) Every nonempty set of real numbers has an upper bound.
- b) A set of real numbers can have at most one upper bound.
- c) Every bounded set of real numbers has exactly one upper bound.
- d) Every bounded set of real numbers has exactly one least upper bound.

THIS TEST CONTINUES ON THE NEXT PAGE!

0/6. Let A and B be nonempty sets and let $f(x, y)$ be an integer valued function on $A \times B$. What is the logical connection between the following two statements?

(P) For every $a \in A$ there exists an element $b \in B$ such that $f(a, b) > 0$.
(Q) There exists an element $b \in B$ such that for every $a \in A$ we have $f(a, b) > 0$.

a) (P) implies (Q).
b) (Q) implies (P).
c) (P) and (Q) are equivalent.
d) None of the statements implies the other.

0/7. Which of the following statements is **false**?
a) The minimum of a set is always an element of the set.
b) The minimum of a set is always a lower bound of the set.
c) The infimum of a set is always an element of the set.
d) The infimum of a set is always a lower bound of the set.

0/8. Which of the following is **impossible** if $a_n \to \infty$ and $b_n \to 0$?
a) $a_n/b_n \to \infty$.
b) $a_n/b_n \to -\infty$.
c) a_n/b_n has a finite limit.
d) a_n/b_n has no finite or infinite limit.

0/9. Which if the following statements is **true**?
a) Every increasing infinite sequence of real numbers has a convergent subsequence.
b) If an infinite sequence of real numbers has no convergent subsequence then the sequence cannot be bounded.
c) Every infinite sequence of real numbers has a convergent subsequence.
d) Every subsequence of a divergent sequence of real numbers is divergent.

0/10. How many subsets A of $\{1, 2, \dots, 10\}$ have the following property:

$$1 \in A \implies 2 \in A$$

a) 256;
b) 512;
c) 768;
d) 1024.

1. ALGEBRA: 4 problems

1/1. Describe geometrically the set of all those complex numbers z on the plane for which $|z-3| = \mathrm{Im}(z+2)$ holds. (Here $\mathrm{Im}\, w$ stands for the imaginary part of the complex number w.)

1/2. Find the value of the following $n \times n$ determinant:

$$\begin{vmatrix} 3 & 3 & 0 & 0 & \dots & 0 & 0 \\ 2 & 3 & 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & 3 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & & 2 & 3 & 1 \\ 0 & 0 & 0 & \dots & \dots & 2 & 3 \end{vmatrix}$$

(Here if a_{ij} denotes the jth entry of the ith row of the matrix then $a_{ii} = 3$ for $1 \le i \le n$, $a_{12} = 3$ and $a_{i(i+1)} = 1$ for $2 \le i \le n-1$, furthermore $a_{i(i-1)} = 2$ for $2 \le i \le n$ and $a_{ij} = 0$ otherwise.)

1/3. Let $p(x) = x^7 + ax^3 + ax^2 + 1 \in \mathbb{R}[x]$. Find all possible values of the parameter $a \in \mathbb{R}$ so that (-1) is at least a double root of $p(x)$.

1/4. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis of the real vector space V. Take the linear transformation $\varphi : V \to V$ defined by $\varphi(\mathbf{b}_1) = \varphi(\mathbf{b}_2) = \varphi(\mathbf{b}_3) = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$.

a) Write the matrix $A = [\varphi]_{\mathcal{B}}$ of the transformation φ relative to the basis $\mathcal{B}$.

b) Find the characteristic polynomial, eigenvalues and eigenvectors of φ.

c) Find an orthonormal basis for every eigenspace of φ.

d) Is there an orthogonal matrix $S \in \mathbb{R}^{3\times 3}$ such that $S^{-1}AS$ is diagonal? (Note that a matrix $B \in \mathbb{R}^{n\times n}$ is orthogonal if the transpose of B is equal to the inverse of B.)

2. GEOMETRY: 4 problems

2/1. Let Σ_1 and Σ_2 be two parallel planes, the distance of which is equal to 8. We know that the intersection of a sphere S with Σ_1 is a circle of radius 5, the intersection of S with Σ_2 is a circle of radius 11. Compute the surface area of S.

2/2. Let O, A, B, C be points in the Euclidean space, not lying in a plane, $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$. Prove that the vector $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ is orthogonal to the plane spanned by the points A, B, and C.

2/3. Show that if a parallelogram is contained in a triangle, then its area is at most half of the area of the triangle.

2/4. Assume that a Cartesian coordinate system is fixed in the 3-dimensional Euclidean space. Compute the reflection of the point $(1, 2, 3)$ in the plane defined by the equation

$$3x + 2y + z = 24.$$

3. COMBINATORICS: 5 problems

3/1. Find the number of integer solutions to the equation $x_1+x_2+x_3 = 25$ under the constraints $0 \le x_i \le 9$ ($\forall\, i \in \{1, 2, 3\}$).

3/2. At an international meeting, each of some 11 countries are represented by 5 diplomats who will be sitting around a round table. Is it possible to seat the diplomats so that for any two countries some of their diplomats are sitting next to each other?

3/3. Alice and Bob play the following game. Alice puts k coins on a 10×10 board in any way she wants; then Bob has to pick ten of these so that no two of the chosen coins are on adjacent squares (we call two squares adjacent if they share an edge or a corner). Bob wins if he succeeds, Alice wins otherwise. What is the least number of coins for which Bob can always win the game, no matter how Alice places them?

3/4. Show that for any non-negative integer n, the equation

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^n = n!$$

holds.

3/5. Show that if we orient the edges of a complete graph on at least two vertices in an arbitrary fashion, then the resulting directed graph must contain a (directed) Hamiltonian path.

4. ANALYSIS: 5 problems

4/1. The sequence (a_n) has the property

$$\lim_{n\to\infty} a_n^n = 2.$$

Prove that

$$\lim_{n\to\infty} a_n^{n+2} = 2.$$

4/2. For any $f : \mathbb{R} \to \mathbb{R}$ and for any $n \in \mathbb{N}$ one defines $f^{\circ n}$ recursively by $f^{\circ 1} := f$ and $f^{\circ n+1} := f \circ f^{\circ n}$. Does there exist some $c \in \mathbb{R}$ such that the polynomial $p(x) = x^{2018} + cx^2$ has the property that for any $a \in \mathbb{R}$ the sequence $a_n := p^{\circ n}(a)$ either diverges to infinity or converges to 0?

4/3. Assume that

$$\sum_{n=1}^{\infty} a_n$$

is convergent. Does this imply the convergence of

$$\sum_{n=1}^{\infty} (-1)^n a_n?$$

4/4. Is the function

$$f(x,y,z) = \begin{cases} \dfrac{xyz}{x^2+y^2+z^2} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0) \end{cases}$$

continuous? Is f differentiable at $(0,0,0)$?

4/5. Calculate

$$\int_0^1 \int_y^1 e^{x^2} \mathrm{d}x \mathrm{d}y.$$

5. PROBABILITY THEORY: 4 problems

5/1. Let X and Y be independent random variables with Poisson distribution. Suppose furthermore that the expectation of X is equal to 3, while the expectation of Y is equal to 5. Find the covariance of $X+Y$ and $X-3Y$.

5/2. We roll a fair die until we get 6 for the second time. Let X_1 be the number of rolls until we get 6 for the first time, and let X_2 be the number of rolls until we get 6 for the second time (e.g. in case of the sequence 364126 we have $X_1 = 2$ and $X_2 = 6$). Find the conditional expectation of X_1 given that $X_2 = 5$.

5/3. Let X be a random variable with the following density function:

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1; \\ \dfrac{3}{14}x^2 & \text{if } 1 \le x < 2; \\ 0 & \text{otherwise.} \end{cases}$$

Find the variance of X.

5/4. Suppose that the random variable X has normal distribution with mean 10 and variance 2. Find the probability of the following event: $|X-9| < 3$.