Helly-type theorems and boxes
Directed studies 1

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1 Introduction

Helly-type theorems state that if a property A holds for any **subfamily** of a family of sets $\mathcal{F}$ that is of a **given finite size** h and property, then some property B holds for the whole **family** $\mathcal{F}$ of **arbitrary finite size** n. An equivalent and often useful formulation provided by negations is that if $\mathcal{F}$ doesn't have property B, then some subfamily of size h doesn't have property A. The minimal number h for which a given Helly-type statement holds will be referred to as the **Helly-number**.
These types of theorems thus provide a structure for algorithms for checking property B for a family of sets whose **running time** is **polinomial** as a function of the **size of the family** as there are at most $\binom{n}{h}$ subfamilies of size h with a given property, which is a polinomial of degree h in n.
The structure of this paper is the following. Section 2 gives an overview of some of the most notable Helly-type theorems that are currently known and provides a more detailed description of two particular types of Helly theorems, namely colorful volume theorems, and box-piercing theorems. Section 3 offers some results in an attempt to combine these two particular directions. Finally, Section 4 presents the proofs of the results.

2 Helly-type theorems

Helly's original statement is about the **emptyness** of the **intersection** of a family of **convex sets in Euclidean space**.
Theorem (Helly). *For a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ if any $(d+1)$-tuple of sets in $\mathcal{F}$ has a non-empty intersection, then all sets in $\mathcal{F}$ have a non-empty intersection.*
Note that here property A and B are the same. This theorem is equivalent to **Radon's theorem** about the convex hull of points in $\mathbb{R}^d$
Lovász and later **Bárány** introduced a property on the **subfamilies**, namely that they be **systems of distinct representatives** of a given substructure that gives a stronger result, the so called **Colorful Helly Theorem**.
Theorem (Colorful Helly Theorem, Lovász, Bárány). *For finite*

families $\mathcal{F}_1, ..., \mathcal{F}_{d+1}$ of convex sets in $\mathbb{R}^d$ if any colorful selection $C_1 \in \mathcal{F}_1, ..., C_{d+1} \in \mathcal{F}_{d+1}$ has a non-empty intersection, then there is a family $\mathcal{F}_i$ such that all sets in $\mathcal{F}_i$ have a non-empty intersection.
The **original Helly** theorem is the **subcase** of this statement when all families are the same. The statement follows from Helly's theorem by considering a **lexicographic ordering** on the points of $\mathbb{R}^d$. More recently **Kalai** and **Meshullum** proved an **extended version** of this theorem which states that not only is there an intersecting family, but it can also be extended by a colorful selection from the other families while still intersecting.
Bárány, **Katschalski** and **Pach** showed a Helly-type theorem about a stronger property B on the family of convex sets. Their **Quantitative Volume Theorem** provides a condition not only for the emptyness of the intersection, but also gives a **lower bound** for the **volume of intersection** of sets.
Theorem (Quantitative Volume Theorem, Bárány, Katschalski, Pach) *For a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ if any $2d$-tuple has an intersection of volume at least 1, then all sets in $\mathcal{F}$ have an intersection of volume at least $c_d == d^{-2d^2}$.*
Note that here property B is weaker than A although both are lower bounds on the volume of the intersection. This is sometimes the case with quantitative volume theorems. Note also that the Helly number is larger than in the original Helly theorem. The constant c_d was later reduced to d^{-2d} by others.

2.1 Colorful Volume Theorems

The results of **Damásdi**, **Földvári** and **Naszódi** combines the conditions of the **colorful** version and the **quantitative** version of Helly's theorem giving a lower bound on the intersection of a family of **convex bodies** (not any convex sets) if there is a common lower bound on the intersection of every colorful selection. In one version the number of families is $d(d+3)/2$ and the **lower bound on the family is the same.**
In the other version there are only $3d$ families and the lower bound has to hold for any colorful selection of size $2d$. The lower bound on the family is $c^{d^2} d^{-5d^2/2}$, however.
These result rely on **John's theorem** about the largest volume ellipsoids contained in convex bodies and the quantitative volume theorem.

2.2 Piercing boxes

Another possible variant of Helly's theorem **generalizes** the notion of **intersection** with the notion of **piercing**.
Definition: A set P **pierces** a family of sets $\mathcal{F}$ if for any set $S \in \mathcal{F}$ there is an element $p \in P$ such that $p \in S$. If $|P| = n$, then $\mathcal{F}$ is **n-pierceable**.
Note that if an intersection of sets is non-emty if and only if it is 1-pierceable.

All previously discussed statements were about families of convex sets. However, there are **no Helly-type theorems** about n-piercing **for all families** of **convex sets** if both property A and B is n-piercability for $n > 1$. For example **Chakraborty** et al. showed that for any constant $h > 0$ there exists a family of **circles** in the plane such that any subfamily of size h is 2-pierceable but the whole family is **not 2-pierceable**.
Danzer and **Grünbaum** showed the **Helly-number** for all possible Helly-type theorems for **n-piercing** families of **axis-parallel boxes in Euclidean space** where both property A and B are n-piercing.
Theorem (Danzer, Grünbaum). *If $h = h(d, n)$ is the smallest positive integer such that for any finite family $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ every h-tuple from $\mathcal{F}$ is n-pierceable implies that $\mathcal{F}$ is n-pierceable then following are the values of h:*

$$\begin{aligned}
h(d,1) &= 2 \\
h(1,n) &= n+1 \\
h(d,2) &= \begin{cases} 3d & : \quad 2 \mid d \\ 3d-1 : & \quad 2 \nmid d \end{cases} \\
h(2,3) &= 16 \\
h(d,n) &= \aleph_0 \quad n \geq 3, (d,n) \neq (2,3)
\end{aligned}$$

Chakraborty, **Ghosh** and **Nandi** combined previous statements and showed an **extended colorful Helly-type theorem for n-piercing intervals and 2-piercing axis-parallel boxes**.
Note that the cases $n = 3, d = 2$ and $n \geq 3, d \geq 3$ are not yet known.
The simple colorful version of this theorem is a trivial consequence of the extended version. Furthermore, the proof of the extended version is not an essential part of the proof as it only follows by adding a last step to the proof after already showing the colorful version.

3 Results

This section presents an attempt at combining the directions in Sections 2.1 and 2.2. Thus, it introduces frameworks which allows for statements about volume that generalize piercing boxes. This is achieved by the notion of punching holes into boxes.

3.1 Punching holes into boxes

Definition: For volume set $\mathcal{V} \subset \mathbb{R}_{>0}$ and enumeration $\nu : \mathcal{V} \to \mathbb{Z}_{>0}$ a family a of d-dimensional boxes $\mathcal{F} = \{\prod_{j=1}^{d}[a_{ij}, b_{ij}] : i \in \mathcal{I}\}$ for some index set $\mathcal{I}$ is

$\mathcal{V}, \nu$**-punchable** if there is a family of d-dimensional boxes $\mathcal{H}$ such that

$$\forall v \in \mathcal{V} \quad \nu(v) = |\{H \in \mathcal{H} : \mathrm{Vol}(H) = v\}| \tag{1}$$

$$\forall B \in \mathcal{F} \quad \exists H \in \mathcal{H} \quad H \subset B \tag{2}$$

If (2) holds for some families of boxes $\mathcal{F}, \mathcal{H}$ then $\mathcal{H}$ **punches** $\mathcal{F}$. If the volume set has 1 element $\mathcal{V} = \{v\}$ and $\nu(v) = n$ and there is a family $\mathcal{H}$ for which (1),(2) hold, then $\mathcal{F}$ is n**-punchable**.

Definition: A family of sets $\mathcal{F}$ is **intersection-connected** if the intersection graph $I_{\mathcal{F}} = (\mathcal{F}, E)$ is connected where $\forall x \neq y \in \mathcal{F}$, $(x, y) \in E \iff x \cap y \neq \emptyset$.

Definition: An intersection-connected family of d-dimensional boxes is **intersection-punchable** if there is a family of d-dimensional boxes $\mathcal{H}$ which **punches** $\mathcal{F}$ such that

$$\forall B \in \mathcal{F} \quad \exists H \in \mathcal{H} \quad \exists B' \in \mathcal{F} \setminus \{B\} \quad H \subset B \cap B' \tag{3}$$

Definition: A family of d-dimensional boxes is n**-sum-**s**-punchable** if there is a family of d-dimensional boxes $\mathcal{H}$ that **punches** $\mathcal{F}$ such that

$$\sum_{H \in \mathcal{H}} \mathrm{Vol}(H) = s \tag{4}$$

$$|\mathcal{H}| = n \tag{5}$$

3.2 Statements

Statement 1: *For a family of intervals* $\mathcal{F} = \{I_i[a_i, b_i] \subset \mathbb{R} : i \in \mathcal{I}\}$ *if any subfamily of* $n+1$*-elements is* n*-punchable, then* $\mathcal{F}$ *is* n*-punchable.*

Statement: *If any translates of a set of* d*-dimensional boxes* $\mathcal{H} = \{A, B\}$ *punches any subfamily of* $3d$ *elements of the family* $\mathcal{F}$ *then* $\mathcal{H}$ *punches* $\mathcal{F}$.

Statement: *For a family of intervals* $\mathcal{F} = \{[a_i, b_i] \subset \mathbb{R} : i \in \mathcal{I}\}$ *if any subfamily of* 3 *elements is intersection-connected and* 2*-sum-*1*-intersection-punchable, then* $\mathcal{F}$ *is* 2*-sum-*1*-(intersection)-punchable.*

Conjecture: *For a family of* d*-dimensional boxes* $\mathcal{F} = \{\prod_{i=1}^{d}[a_i, b_i] \subset \mathbb{R} : i \in \mathcal{I}\}$ *if any subfamily of* $n+1$ *elements is intersection-connected and* n*-sum-*1*-intersection-punchable, then* $\mathcal{F}$ *is* n*-sum-*1*-(intersection)-punchable.*

Observation: *If any* $2d$ *element subfamily of a family of* d*-dimensional boxes is* 1*-punchable, then* $\mathcal{F}$ *is* 1*-punchable.*

Statement 2: *For any dimension* d *there is a family* $\mathcal{F}$ *of* d*-dimensional boxes such that any* $3d$*-tuple is* 2*-punchable, but* F *is only* $\{\varepsilon\}, 2$*-punchable for any* $\varepsilon > 0$.

Corollary: *In any Helly-type theorem about 2-punching boxes, the Helly number has to be at least* $3d+1$.

Conjecture: *For a family of* d*-dimensional boxes* $\mathcal{F} = \{\prod_{i=1}^{d}[a_i, b_i] \subset \mathbb{R}^d : i \in \mathcal{I}\}$ *if any subfamily of* $4d$*-elements is* 2*-punchable, then* $\mathcal{F}$ *is* 2*-punchable.*

4 Proofs

Proof of Statement 1:
Observation: *If $A, B \subset \mathbb{R}^d$ are convex sets, then their Minkowski-difference $A - B$ is also a convex set.*
Observation: *If $A, B, C \subset \mathbb{R}^d$ then $C + t \subset A \cap B$ for some $t \in \mathbb{R}^d$ if and only if $A - C \cap B - C \neq \emptyset$ where $S - T$ denotes the Minkowski-difference.*
Proof: $A - C$ equals the set of vectors v such that $C + v \subset A$.
Observation: *All intervals of volume 1 are translates of each other.*
Let $I = [0, 1]$, then any tuple $I_1, ..., I_{n+1}$ is n-punchable if and only if $I_i - I, ..., I_{n+1} - I$ is n-piercable. The theorem follows thus from the theorem of Danzer and Grünbaum about n-piercing intervals. ■
Proof of Statement 2:
The following families of boxes of size $4d$ have the given property.
For d dimensions let $B_{ij} = \prod_{k=1}^{d} I_k$ for $1 \leq i \leq d$, $1 \leq j \leq 4$, where $I_k = [-2, 2] = I$ for $k \neq i$ and $I_i = \begin{cases} [-2, -1 + \varepsilon/2] : j = 1 \\ [-1 - \varepsilon/2, 0] : j = 2 \\ [0, 1 + \varepsilon/2] : j = 3 \\ [1 - \varepsilon/2, 2] : j = 4 \end{cases}$
Then $\mathcal{F} = \{cB_{ij} : 1 \leq i \leq d, 1 \leq j \leq 4\}$ where $c = \frac{1}{(\varepsilon^{d-1}(1+\varepsilon/2))^{1/d}}$.
Claim: *Any subfamily $\mathcal{F}' \subset \mathcal{F}$ of size $3d$ is 2-punchable.*
Proof: Since $|\mathcal{F}| = 4d$ and $|\mathcal{F}' = 3d|$ there is either a) an i for which only 2 B_{ij_1}, B_{ij_2} boxes are in $\mathcal{F}'$ or b) there are 3 for any i. Since all boxes in $\mathcal{F}$ are punched by the intersection of $\mathcal{F}$ in case a) $\mathcal{F}' \setminus \{B_{ij_1}, B_{ij_2}\}$ are also punched by two ε-boxes of the intersection of $\mathcal{F}$. By exteding these to I_{ij_1} and I_{ij_2} along the coordinate axis i, we get two punching boxes of size 1. In case b) there is a box B_{ij} for every i that does not intersect any other $B_{ij'}$ with the same i. Then given two punching boxes from the intersection of $\mathcal{F}$, one can be extended along coordinate axis i for B_{ij} of the previous property and for the other also punches another $B_{i'j'}$ with this property, so the box can be extended along the i' axis for the other box. Thus we get two punching boxes of size 1.
$\mathcal{F}$ can only be punched by 2 boxes of size $(c\varepsilon)^d = \frac{\varepsilon}{1+\varepsilon} < \varepsilon$ for any $\varepsilon > 0$. ■

References

[1] Damásdi, G., Viktória Földvári, V. & Naszódi,M. (2020). Colorful Helly-type theorems for the volume of intersections of convex bodies. Journal of Combinatorial Theory.

[2] Chakraborty. S., Ghosh, A. & Nandi, S. (2022). Coloful Helly Theorem for Piercing Boxes with Two Points.