Superlinear convergence of iterative methods for elliptic PDEs and systems

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Outline

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The problem

- Let $d \geq 2$, $p > 2$.
- Let $\Omega \subset \mathbb{R}^d$ be a bounded domain.
- We consider the elliptic problem

$$\begin{cases} -\mathrm{div}(G\nabla u) + \eta u = g, \\ u_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\eta = \eta(x)$ and G is a constant matrix.

Construction of the discretization

- Let $V_h \subset H_0^1(\Omega)$ be a FEM subspace. We look for u_h in V_h:

$$\int_\Omega (G\nabla u_h \cdot \nabla v + \eta u_h v) = \int_\Omega gv, \quad v \in V_h. \tag{1.2}$$

The corresponding linear algebraic system:

$$\underbrace{\mathbf{B}_h}_{(\mathbf{G}_h+\mathbf{D}_h)} \mathbf{c} = \mathbf{g}_h.$$

Preconditioned form:

$$\underbrace{\mathbf{G}_h^{-1}\mathbf{B}_h}_{(\mathbf{I}_h+\mathbf{G}_h^{-1}\mathbf{D}_h)} \mathbf{c} = \underbrace{\tilde{\mathbf{g}}_h}_{\mathbf{G}_h^{-1}\mathbf{g}_h}. \tag{1.3}$$

- Preconditioned conjugate gradient method (PCGM)= CGM for (1.3).

PCGM Algorithm

- Let $u_0 \in H$ arbitrary, $\rho_0 = \mathbf{B}_h u_0 - g$, $\mathbf{G}_h p_0 = \rho_0$, $r_0 = \rho_0$.
- For $k \in \mathbb{N}$:

$$\begin{cases} u_{k+1} = u_k + \alpha_k p_k, \\ r_{k+1} = r_k + \alpha_k \mathbf{G}_h^{-1} \mathbf{B}_h p_k, \\ p_{k+1} = r_{k+1} + \beta_k p_k. \end{cases}$$

- Auxiliary problems:

$$\mathbf{G}_h z_k = \mathbf{B}_{\boldsymbol{h}} p_k.$$

They can be solved easily with fast solvers due to the special structure of $\mathbf{G}_h$, [9], [5].

Superlinear convergence of the PCGM

- We study the error vectors $e_k = c - c_k$.
- Let $\mathbf{A}_h = (\mathbf{I}_h + \mathbf{G}_h^{-1}\mathbf{D}_h)$. It is known [3] that

$$\left(\frac{\|e_k\|_{\mathbf{A}_h}}{\|e_0\|_{\mathbf{A}_h}}\right)^{1/k} \le \underbrace{\frac{2\|\mathbf{A}_h^{-1}\|}{k}\sum_{j=1}^{k}\left|\lambda_j(\mathbf{G}_h^{-1}\mathbf{D}_h)\right|}_{\text{A mesh dependent bound}}. \qquad (1.4)$$

Objectives

- Derive mesh-independent superlinear convergence of the preconditioned CGM.
- Estimate the rate of superlinear convergence.
- Extend the results of [8] from $\eta \in C(\overline{\Omega})$ to $\eta \in L^q(\Omega)$.

Assumptions

(i) $G \in \mathrm{L}^{\infty}_{symm}(\overline{\Omega}, \mathbb{R}^{d\times d})$ satisfies

$$G(x)\xi \cdot \xi \geq m|\xi|^2$$

for all $\xi \in \mathbb{R}^d$, with some $m > 0$ independent of ξ.

(ii) There exists $2 < p < \frac{2d}{d-2}$:

$$\eta \in \mathrm{L}^{p/(p-2)}(\Omega).$$

Main theorem

Theorem 1 (Superlinear convergence rate estimation)

Let $2 < p < \frac{2d}{d-2}$. *Then there exists* $\boldsymbol{C} > 0$ *such that*

$$\left(\frac{\|e_k\|_A}{\|e_0\|_A}\right)^{\frac{1}{k}} \leq \frac{\boldsymbol{C}}{k^\alpha} \to 0, \qquad as\ k \to \infty, \tag{2.5}$$

where $\alpha = \frac{1}{d} - \frac{1}{2} + \frac{1}{p}$.

Sketch of the proof

- We define the operators

$$Su \equiv -\mathrm{div}(G\nabla u), \quad u \in D \quad \text{and} \quad Qu \equiv \eta u, \quad u \in \mathrm{H}_0^1(\Omega).$$

- Energy space: $H_S = \mathrm{H}_0^1(\Omega)$ with

$$\langle u, v\rangle_S = \int_\Omega G\nabla u \cdot \nabla v. \tag{2.6}$$

It can be proved that there exists a unique operator $Q_S \in \mathcal{B}(H_s)$ such that

$$\langle Q_S u, v\rangle_S = \langle Qu, v\rangle$$

for all $u, v \in H_S$.

Lemma 1

There exists $C > 0$ such that

$$\|Q_S v\|_{H_S} \leq C\|v\|_{L^p(\Omega)}, \quad \forall v \in H_S. \tag{2.7}$$

Altogether, Q_S is compact and self-adjoint in H_S.

Proposition 1

Let $A = I + Q_S$. For any $k = 1, 2, \ldots, n$

$$\sum_{j=1}^{k} |\lambda_j(\mathbf{G}_h^{-1}\mathbf{D}_h)| \leq \sum_{j=1}^{k} \lambda_j(Q_S). \tag{2.8}$$

Moreover,

$$\left(\frac{\|e_k\|_A}{\|e_0\|_A}\right)^{1/k} \leq 2\|A^{-1}\|\frac{1}{k}\sum_{j=1}^{k} \lambda_j(Q_S). \tag{2.9}$$

Now we wish to get a rate estimation from (2.9).

Useful results

1. Let $\lambda_n = \lambda_n(Q_S)$. Since Q_S is a compact self-adjoint operator in H_S, we have the characterization, [7, Ch.6, Th.1.5]:

   $$\lambda_n(Q_S) = \min\{\|Q_S - L_{n-1}\| \,/\, L_{n-1} \in \mathcal{L}(H_S), \mathrm{rank}(L_{n-1}) \leq n-1\}.$$

2. Let $a_n(\mathcal{I})$ denote the approximation numbers of the compact embedding $\mathcal{I}\colon \mathrm{H}_0^1(\Omega) \mapsto \mathrm{L}^p(\Omega)$, defined as

   $$a_n(\mathcal{I}) = \min\{\|\mathcal{I} - L_{n-1}\| / \, L_{n-1} \in \mathcal{L}(\mathrm{H}_0^1(\Omega), \mathrm{L}^p(\Omega)), \mathrm{rank}(L_{n-1}) \leq n-1\}.$$

3. From [6] we have the estimation

   $$a_n(\mathcal{I}) \leq \hat{C} n^{-\alpha}, \quad \alpha = \frac{1}{d} - \frac{1}{2} + \frac{1}{p},$$

   for some constant $\hat{C} > 0$.

Proposition 2

There exists $C > 0$, such that for all $n \in \mathbb{N}$,

$$\lambda_n(Q_S) \leq C a_n(\mathcal{I}). \tag{2.10}$$

Proposition 3

There exists $C > 0$ such that

$$\frac{1}{k}\sum_{n=1}^{k} \lambda_n(Q_S) \leq C\frac{1}{k^\alpha}.$$

Finally, by (2.9), the theorem is proved. □

Extension to elliptic systems

Systems of PDE's:

$$\begin{cases} -\Delta u_i + \eta_{i1} u_1 + \dots \eta_{is} u_s = g_i, \\ u_i|_{\partial\Omega} = 0, \quad (i = 1, \dots, s), \end{cases} \tag{2.11}$$

where $\boldsymbol{H} = \{\eta_{ij}\}_{i,j=1}^{s} : \Omega \to \mathbb{R}^{s\times s}_{symm}$ such that:

$$\forall i,j \in \{1, \dots, s\} : \quad \eta_{ij} \in \mathrm{L}^{p/(p-2)}(\Omega).$$

Main result

Theorem 1 holds for (2.11) as well.

Sketch of the proof

- We work with the space $L^p(\Omega)^s$ with the norm

$$\|u\|_{L^p(\Omega)^s} = \left(\sum_{j=1}^{s} \|u_j\|^2_{L^p(\Omega)} \right)^{1/2}, \quad u = (u_1, \ldots, u_s) \in L^p(\Omega)^s.$$

- Let $H = L^2(\Omega)^s$. Let $u = (u_1 \ldots u_s) \in D = (H^1_0(\Omega) \cap H^2(\Omega))^s$, we define the operators

$$Su = \begin{pmatrix} -\Delta u_1 \\ . \\ . \\ . \\ -\Delta u_s \end{pmatrix}, \qquad Qu = \boldsymbol{H}u, \qquad u \in H^1_0(\Omega)^s. \qquad (2.12)$$

- Energy space: $H_S = \mathrm{H}_0^1(\Omega)^s$ with

$$\langle u, v \rangle_S = \sum_{i=1}^{s} \int_\Omega \nabla u_i \nabla v_i, \quad \|u\|_{H_S}^2 = \sum_{i=1}^{s} \int_\Omega |\nabla u_i|^2.$$

- There exists a unique operator $Q_S \in \mathcal{B}(H_s)$ such that

$$\langle Q_S u, v \rangle_S = \int_\Omega \sum_{i,j=1}^{s} \eta_{ij} u_j v_i. \qquad (2.13)$$

Extensions of (2.7) and (2.10), respectively:

Lemma 3

There exists $C > 0$ such that

$$\|Q_S v\|_{H_S} \leq C\|v\|_{(\mathrm{L}^p(\Omega))^s}, \quad \forall v \in H_S. \tag{2.14}$$

Altogether, Q_S is compact and self-adjoint in H_S.

Lemma 4

There exists $C > 0$, such that for all $n \geq s+1$,

$$\lambda_n(Q_S) \leq C\left[\frac{n-1}{s}\right]^{-\alpha}. \tag{2.15}$$

Finally, from these results the theorem can be deduced. □

Advantages of the preconditioner

The auxiliary problem $\mathbf{S}_h w_k = \mathbf{Q}_h p_k$ for the PCGM becomes

$$\begin{cases} -\Delta(w_k)_1 & = \sum_{j=1}^{s} \eta_{1j}(p_k)_j, \\ -\Delta(w_k)_2 & = \sum_{j=1}^{s} \eta_{2j}(p_k)_j, \\ & . \\ & . \\ & . \\ -\Delta(w_k)_s & = \sum_{j=1}^{s} \eta_{sj}(p_k)_j, \\ (w_i)|_{\partial\Omega} & = 0, \quad \forall i = 1, \dots, s. \end{cases}$$

These equations are independent of one another, hence they can be solved in parallel.

A numerical example

Let us solve the following PDE numerically:

$$\begin{cases} -\Delta u + \eta u = f, & \text{in } \Omega = [0,1]^2, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{3.16}$$

Here $\eta \in \mathrm{L}^{\frac{p}{p-2}}(\Omega)$ is defined as

$$\eta(x,y) = (x^2 + y^2)^{-\beta}, \qquad 0 < \beta < \frac{p-2}{p}$$

and

$$f(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y) + \eta(x,y)\sin(\pi x)\sin(\pi y).$$

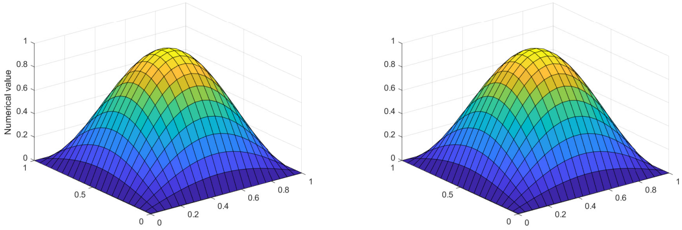
The exact solution of (3.16) is

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

- Apply FEM with stepsize $h = \frac{1}{N+1}$. Algebraic system:

$$(\mathbf{G}_h + \mathbf{D}_h)\mathbf{c} = \mathbf{g}_h. \tag{3.17}$$

- Choose $\mathbf{G}_h$ as a preconditioner and apply PCGM.



(a) Numerical solution with $N = 20$.

(b) Exact solution

Figure: Graphs of the numerical and exact solution with $\beta = 1/4$.

To measure the error of the PCGM, we use the discrete Sobolev norm

$$|e|_{1,h} = \langle -\Delta_h e, e \rangle^{\frac{1}{2}} \quad \left(e \in \mathbb{R}^d \right).$$

k	$|e_k|_{1,h}$
1	0.08720185611160
2	0.00163282922100
3	0.00007900091053
4	0.00000007687816
5	0.00000000030254
6	0.00000000000164
7	0.00000000000001

Table: Error obtained with PCGM with $N = 20$, $\beta = 1/4$.

Recall our main result:

Superlinear convergence rate

There exists $\boldsymbol{C} > 0$ such that

$$\left(\frac{\|e_k\|_A}{\|e_0\|_A}\right)^{\frac{1}{k}} \leq \frac{\boldsymbol{C}}{k^{\alpha}} \to 0, \text{ as } k \to \infty.$$

Here $\alpha = \frac{1}{d} - \frac{1}{2} + \frac{1}{p}$.

To test this result, we study the values of

$$\delta_k = \left(\frac{|e_k|_{1,h}}{|e_0|_{1,h}}\right)^{\frac{1}{k}} k^{\alpha} \underbrace{\leq \boldsymbol{C}}_{\textit{Expectation}} .$$

We performed several runs for different values of α, β. Notice that the theorem holds when $\alpha < \frac{1-\beta}{2}$.

k	$\beta = 1/4$, $\alpha = 0.2$	$\beta = 1/3$, $\alpha = 0.2$	$\beta = 1/2$, $\alpha = 0.2$	$\beta = 2/3$, $\alpha = 0.1$	$\beta = 3/4$, $\alpha = 0.08$
1	0.0078	0.0087	0.0116	0.0171	0.0218
2	0.0155	0.0158	0.0194	0.0232	0.0261
3	0.0130	0.0135	0.0186	0.0231	0.0269
4	0.0142	0.0142	0.0183	0.0213	0.0242
5	0.0131	0.0128	0.0169	0.0197	0.0227
6	0.0130	0.0122	0.0152	0.0164	0.0187
7	0.0143	0.0122	0.0141	0.0150	0.0173

Table: Values of δ_k for different α's and β's for a fixed mesh size $N = 40$.

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Thank you for your attention!

Our result gives a mesh-independent bound.

k	$N = 20$	$N = 40$	$N = 80$
1	0.0105	0.0078	0.0057
2	0.0173	0.0155	0.0134
3	0.0137	0.0130	0.0120
4	0.0149	0.0142	0.0133
5	0.0132	0.0131	0.0126
6	0.0131	0.0130	0.0125
7	0.0134	0.0143	0.0155

Table: Values of δ_k for different mesh sizes with $\beta = 1/4$, $\alpha = 0.2$.