Current progress on the single-coordinate translation-invariant of product measure problem

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1 Introduction

Let $(X, \mathcal{A}, \mu)$ be a measure space with σ-algebra $\mathcal{A}$ and measure μ, and $(\mathbb{R}, \mathcal{B}, \nu)$ be a real measure space with σ-algebra $\mathcal{B}$ and the Lebesgue measure ν. Denote their product measure space by $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$, where the product measure is arbitrary. We shall fix these measure spaces throughout the report.

The Lebesgue measure is know to be translation-invariant. One question we may ask is whether the product measure $\mu \times \nu$ inherits this property in the sense that any shift of a measurable set $B \in \mathcal{A} \otimes \mathcal{B}$ along the real axis does not alter the measure. Formally, we conjecture

1.1 Conjecture. *Let the product space $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ and a set $B \in \mathcal{A} \otimes \mathcal{B}$ be given. For any $c \in \mathbb{R}$, define the vertical shift of B by c as the set*

$$B + c \coloneqq \{(x, y + c) : (x, y) \in B\} \in \mathcal{A} \otimes \mathcal{B}.$$

Then, $\mu \times \nu\,(B + c) = \mu \times \nu\,(B)$.

If the measure space $(X, \mathcal{A}, \mu)$ is σ-finite, then the conjecture holds as a consequence of Fubini-Tonelli's theorem. As for the non-σ-finite case, we are currently considering different tools such as Borel code and ultraproduct construction. We decided to explore the latter option in this semester.

In this report, we will first show that the vertical shift $B + c$ of B is measurable. Then, we will introduce relevant notions for the ultraproduct construction as well as reflections on our current approach.

2 The vertical shift of a measurable set is measurable

The conjecture has little meaning if there is a measurable set $B \in \mathcal{A} \otimes \mathcal{B}$ and a number $c \in \mathbb{R}$ such that $B + c \notin \mathcal{A} \otimes \mathcal{B}$. By transifinite induction, we will see that the vertical shift $B + c$ of B is measurable. The arguments of the proof has the same flavour to that of the construction of a generated σ-algebra in the following lemma.

2.1 Lemma. *Let X be a set and $\{\emptyset, X\} \subseteq \mathcal{C} \subseteq \mathcal{P}(X)$ be a family of (generating) sets. Let α be an ordinal and λ be a limit ordinal. Define*

1. *$\mathcal{F}_0 \coloneqq \mathcal{C}$;*

2. $\mathcal{F}_{\alpha+1} \coloneqq \mathcal{F}_\alpha \cup \left\{\overline{F} : A \in \mathcal{F}_\alpha\right\} \cup \left\{\bigcup_{n\in\mathbb{N}} F_n : F_n \in \mathcal{F}_\alpha\right\}$ *and*

3. $\mathcal{F}_\lambda \coloneqq \displaystyle\bigcup_{\alpha<\lambda} F_\alpha$.

Then, $\mathcal{F}_{\omega_1}$ *is the* σ*-algebra generated by* $\mathcal{C}$.

We refer interested readers to the proof of this lemma in [1].

2.2 Theorem. *Let the product space* $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ *and a set* $B \in \mathcal{A} \otimes \mathcal{B}$ *be given. For any* $c \in \mathbb{R}$, *we have* $B + c \in \mathcal{A} \otimes \mathcal{B}$.

Proof. Let $\mathcal{F}_0 \coloneqq \mathcal{C}$ be the family of measurable rectangles

$$\mathcal{C} \coloneqq \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}.$$

Let α be an ordinal, and λ be a limit ordinal. Define $\mathcal{F}_{\alpha+1}$ and $\mathcal{F}_\lambda$ as in lemma 2.1.

It is clear that $F \in \mathcal{A} \otimes \mathcal{B}$ for all $F \in \mathcal{F}_0 = \mathcal{C}$. Suppose that for an ordinal α, every $F \in \mathcal{F}_\alpha$ is such that $F + c \in \mathcal{A} \otimes \mathcal{B}$.

- It holds that $\overline{F} + c = \overline{(F + c)}$. Consequently, $\overline{F} + c \in \mathcal{A} \otimes \mathcal{B}$.
- Let $F_n \in \mathcal{F}_\alpha$ where $n \in \mathbb{N}$. With $\displaystyle\bigcup_{n\in\mathbb{N}} F_n + c = \bigcup_{n\in\mathbb{N}} (F_n + c)$, we obtain $\displaystyle\bigcup_{n\in\mathbb{N}} F_n + c \in \mathcal{A} \otimes \mathcal{B}$.

Thus, every $F \in \mathcal{F}_{\alpha+1}$ is such that $F + c \in \mathcal{A} \otimes \mathcal{B}$.

Recall that $\lambda \leq \omega_1$ is a limit ordinal. Assume $F + c \in \mathcal{A} \otimes \mathcal{B}$ where $F \in \mathcal{F}_\alpha$ holds for all $\alpha < \lambda$. If $F \in \mathcal{F}_\lambda = \displaystyle\bigcup_{\alpha<\lambda} F_\alpha$, then there exists an ordinal $\alpha < \lambda$ such that $F \in \mathcal{F}_\alpha$. Indeed, $\overline{F} \in \mathcal{F}_{\alpha+1}$ and $\displaystyle\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{F}_{\alpha+1}$. So, $\overline{F}+c \in \mathcal{A}\otimes\mathcal{B}$ and $\displaystyle\bigcup_{n\in\mathbb{N}} F_n+c \in \mathcal{A}\otimes\mathcal{B}$ as shown previously. □

3 The ultraproduct construction

Our current sketch of the proof focuses on working with the measure space as an embedding into the ultraproduct structure, which is formed under an equivalence relation defined by an ultrafilter. There remains some questions regarding the generality of this construction which we shall discuss at the end of this report. We will now formally define the relevant notions, starting from the definition of filter.

3.1 Definition. Let X be a non-empty set and $\mathcal{P}(X)$ be its power set. Then, the non-empty family $\mathcal{F} \subsetneq \mathcal{P}(X)$ is called a filter on X if

- for every sets $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$, and
- for every $B \in \mathcal{P}(X)$ and $A \in \mathcal{F}$, if $A \subseteq B$ then $B \in \mathcal{F}$.

We say that $\mathcal{F}$ is a proper filter if $\mathcal{F} \neq \mathcal{P}(X)$, which is equivalent to $\emptyset \notin \mathcal{F}$. One variant of proper filter of our interest is the ultrafilter defined as follows.

3.2 Definition. Let X be a non-empty set and $\mathcal{P}(X)$ be its power set. Then, the non-empty family $\mathcal{F} \subsetneq \mathcal{P}(X)$ is called an ultrafilter on X if

- $\emptyset \notin \mathcal{F}$;

- for every sets $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$;
- for every $B \in \mathcal{P}(X)$ and $A \in \mathcal{F}$, if $A \subseteq B$ then $B \in \mathcal{F}$, and
- for any $A \in \mathcal{P}(X)$, we have that either $A \in \mathcal{F}$ or $X \backslash A \in \mathcal{F}$.

The ultrafilter captures the notion of "large" sets, which is possible to observed more concretely through a set function $m : \mathcal{P}(X) \to \{0, 1\}$ assigning 1 to the sets in the ultraproduct and 0 otherwise. One can check that this set function yields a finitely additive measure. Moreover, we can prove that the ultrafilter is the maximal proper filter. We may obtain the ultrafilter extending a filter via Zorn's lemma.

3.3 Definition. Let I be a non-empty index set. Let X_i be sets, $i \in I$, and $\mathcal{F}$ be an ultrafilter on I. Let $u, v : I \to \bigcup_{i\in I} X_i$ be elements of the space $\prod_{i\in I} X_i$. We define the ultraproduct of $\{X_i\}_{i\in I}$ under $\mathcal{F}$ to be the space $\prod_{i\in I} X_i$ under the equivalence relation

$$u \equiv v \iff \{i : u(i) = v(i)\} \in \mathcal{F}.$$

We denote the ultraproduct by $\prod_{i\in I} X_i/\mathcal{F}$.

In other words, the equivalence relation above suggests that two elements u, v of the ultraproduct space $\prod_{i\in I} X_i/\mathcal{F}$ are the same if they coincide pointwise almost everywhere , according to the finitely additive measure $m : \mathcal{P}(X) \to \{0, 1\}$ defined previously.

4 The current progress using the ultraproduct structure

To sketch our current direction of the conjecture, consider a measure space $(X, \mathcal{A}, \mu)$. The family of finite measure sets $I := \{X_i \subseteq X : \mu(X_i) < \infty\}$ generates a filter, which can be extended to an ultrafilter $\mathcal{F}$. To simplify the notation, we give the identification $i \equiv X_i$. Define the ultraproduct $\prod_{i\in I} X_i/\mathcal{F}$ whose coordinate space X_i is equipped with the measure μ restricted to X_i for $i \in I$. We shall limit our attention to the family of subsets called decomposable sets, which are closed under complement.

4.1 Definition. Let $X \subseteq \prod_{i\in I} X_i/\mathcal{F}$ be a subset. We say that X is decomposible iff for all $i \in I$, there exists $A_i \subseteq X_i$ such that $X = \prod_{i\in I} A_i/\mathcal{F}$.

If the ultrafilter assumes κ-regular property, where κ is an infinite cardinal, then the family of decomposable sets is closed under κ-union.

4.2 Definition. Let I a non-empty index set. Let κ be an infinite cardinal, and $\mathcal{F}$ be an ultrafilter. We say $\mathcal{F}$ is a κ-regular iff $E \subseteq \mathcal{F}$ where $|E| = \kappa$ is such that for all $i \in I$ we have $\{e \in E : i \in e\}$ is finite.

This shall allow us to define σ-algebra in the ultraproduct. It is possible to equip a measure on this σ-algebra with a help of the ultralimit defined as

4.3 Definition. Let $\mathcal{F}$ be an ultrafilter on I. Let $\{a_i\}_{i\in I} \subset \mathbb{R}$ be a sequence of real numbers. We say that a is the ultralimit, denoted by $a \coloneqq \lim_{\mathcal{F}} a_i$ if for every $\varepsilon > 0$ we have

$$\{i \in I : |a_i - a| < \varepsilon\} \in \mathcal{F}.$$

For a decomposable set $X = \prod_{i\in I} A_i/\mathcal{F}$, we then assign the measure using the ultralimit as

$$\mu(X) = \lim_{\mathcal{F}} \mu_i(X_i).$$

Our hope is to induce translation invariant property in the finite product measure space to the whole product space via this construction. What we also need is to show that we can truly embed our product measure space in the ultraproduct. This establishes the plan of our work in the next semester.

5 Remarks on the current approach

In general, we believe that there will be no problem for the case where $(X, \mathcal{A}, \mu)$ is σ-finite. However, there are little technical issues to be examined when working with a non-σ-finite measure space. For example, it is possible to ask whether the ultraproduct construction introduce would be appropriate if $(X, \mathcal{A}, \mu)$ contains an atom of infinite measure since our ultrafilter was built from the sets of finite measure. If not, we hope to explore whether the issue can be addressed separately or with another approach.

References

[1] David Ross (2013) *Lecture notes: Transfinite induction for measure theory*, `https://math.hawaii.edu/~ross/Math671/lecnotes_tfinduction.pdf`, Last visited: 16 December 2022.

[2] C.C. Chang and H.J. Keisler (1990) *Model Theory*, Dover Publications, 3rd ed.