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Current progress on the single-coordinate translation-invariant of product measure problem

Author: Nicha Khenkhok
Supervisor: Gábor Sági, PhD.

December 23, 2022

Introduction

Product measure space

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces.
A product measure space is the space $X \times Y$ equipped with

- the σ-algebra $\mathcal{A} \otimes \mathcal{B}$ generated by the set $\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$,
- a product measure $\lambda : \mathcal{A} \otimes \mathcal{B} \to \mathbb{R}_0^+$.

Product measure

A measure $\lambda : \mathcal{A} \otimes \mathcal{B} \to \mathbb{R}_0^+$ is a product measure of μ and ν if for all $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$\lambda(A \times B) = \mu(A)\nu(B).$$

Distinct product measures on the same space

Disclaimer: product measure is not necessarily unique. Let $E \in \mathcal{A} \otimes \mathcal{B}$, we define

The primitive product measure:

$$\pi(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n) : \mathcal{A}_n \in \mathcal{A}, B_n \in \mathcal{B}, E \subseteq \bigcup_{n=1}^{\infty} A_n \times B_n \right\}.$$

The completely locally determined (c.l.d) product measure:

$$\rho(E) = \sup \left\{ \pi(E \cap (A \times B)) : \mathcal{A} \in \mathcal{A}, B \in \mathcal{B}; \mu(A), \nu(B) < \infty \right\}.$$

DISTINCT PRODUCT MEASURES ON THE SAME SPACE

Suppose that

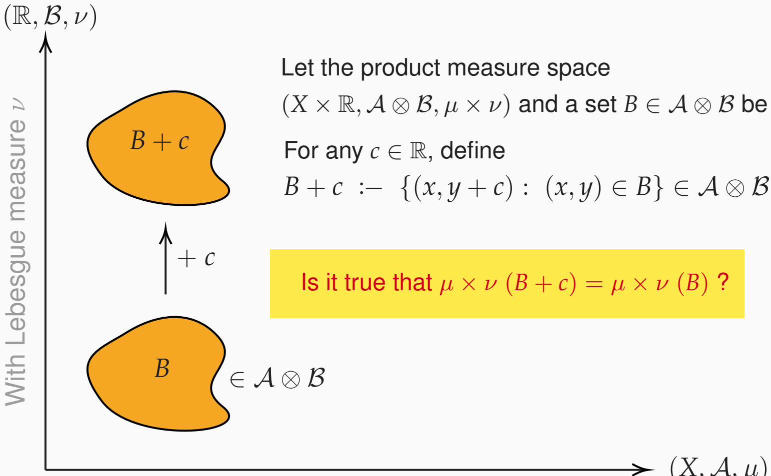
- $X, Y = [0,1]$;
- $\mathcal{A} =$ Lebesgue σ-algebra, $\mathcal{B} = \mathcal{P}([0,1])$;
- $\mu =$ Lebesgue measure, $\nu =$ counting measure.

Consider the set $\Delta = \{(x,x) : x \in [0,1]\}$ in $\mathcal{A} \otimes \mathcal{B}$

$$\Delta = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \left[\frac{k}{n}, \frac{k+1}{n}\right] \times \left[\frac{k}{n}, \frac{k+1}{n}\right]$$

Then, the primitive product measure gives $\pi(\Delta) = +\infty$ and the c.l.d measure gives $\rho(\Delta) = 0$.

Introduction



Let the product measure space $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ and a set $B \in \mathcal{A} \otimes \mathcal{B}$ be given.

For any $c \in \mathbb{R}$, define

$$B + c \ :- \ \{(x, y + c) : \ (x, y) \in B\} \in \mathcal{A} \otimes \mathcal{B}.$$

Is it true that $\mu \times \nu \ (B + c) = \mu \times \nu \ (B)$?

Preliminary check

Preliminary check

We need that any vertical translate $B + c$ of B is in the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$. Its proof utilises ideas from ...

Construction of a generated σ-algebra

Let X be a set and $\{\emptyset, X\} \subseteq \mathcal{C} \subseteq \mathcal{P}(X)$ be a family of (generating) sets. Let α be an ordinal and λ be a limit ordinal. Define

1. $\mathcal{F}_0 := \mathcal{C}$;
2. $\mathcal{F}_{\alpha+1} := \mathcal{F}_\alpha \cup \left\{\overline{F} : A \in \mathcal{F}_\alpha\right\} \cup \left\{\bigcup_{n\in\mathbb{N}} F_n : F_n \in \mathcal{F}_\alpha\right\}$ and
3. $\mathcal{F}_\lambda := \bigcup_{\alpha<\lambda} F_\alpha$.

Then, $\mathcal{F}_{\omega_1}$ is the generated by $\mathcal{C}$.

Ultraproduct construction

Ultrafilter and ultraproduct

Ultrafilter

Let X be a non-empty set and $\mathcal{P}(X)$ be its power set. Then, the non-empty family $\mathcal{F} \subsetneq \mathcal{P}(X)$ is called an **ultrafilter** on X if

- $\emptyset \notin \mathcal{F}$;
- for every sets $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$;
- for every $B \in \mathcal{P}(X)$ and $A \in \mathcal{F}$, if $A \subseteq B$ then $B \in \mathcal{F}$, and
- for any $A \in \mathcal{P}(X)$, we have that either $A \in \mathcal{F}$ or $X \backslash A \in \mathcal{F}$.

Our ultrafilter $\mathcal{F}$ will be built from the family of finite measure sets in the product measure space.

Ultrafilter and ultraproduct

Ultraproduct

Let I be a non-empty index set. Let X_i be sets, $i \in I$, and $\mathcal{F}$ be an ultrafilter on I. Let $u, v : I \to \bigcup_{i\in I} X_i$ be elements of the space $\prod_{i\in I} X_i$. We define the **ultraproduct** of $\{X_i\}_{i\in I}$ under $\mathcal{F}$ to be the space $\prod_{i\in I} X_i$ under the equivalence relation

$$u \equiv v \iff \{i : u(i) = v(i)\} \in \mathcal{F}.$$

We denote the ultraproduct by $\prod_{i\in I} X_i/\mathcal{F}$.

Each X_i will be finite measure sets in $\mathcal{A} \otimes \mathcal{B}$ with the measure $(\mu \times \nu)_i$ obtained through the restriction of $\mu \times \nu$ to X_i.

The current progress via the ultraproduct structure

σ-ALGEBRA AND MEASURE

QUESTION 1.
What is the copy of our σ-algebra in the embedded measure space?

Plan: work with decomposable sets and κ-regularity.

DECOMPOASBLE SETS

Let $X \subseteq \prod_{i\in I} X_i/\mathcal{F}$ be a subset. We say that X is **decomposable** iff for all $i \in I$, there exists $A_i \subseteq X_i$ such that $X = \prod_{i\in I} A_i/\mathcal{F}$.

The family of decomposable sets are closed under complementation.

σ-ALGEBRA AND MEASURE

κ-REGULARITY

Let I a non-empty index set. Let κ be an infinite cardinal, and $\mathcal{F}$ be an ultrafilter. We say $\mathcal{F}$ is **κ-regular** iff there exists a subfamily $E \subseteq \mathcal{F}$ where $|E| = \kappa$ is such that for all $i \in I$ we have $\{e \in E : i \in e\}$ is finite.

If our ultrafilter is κ-regular, then we have strong control on the κ-complete Boolean algebra generated by the family of decomposable sets.

σ-ALGEBRA AND MEASURE

QUESTION 2.
How can we define the measure within the ultraproduct construction?

ULTRALIMIT

Let $\mathcal{F}$ be an ultrafilter on I. Let $\{a_i\}_{i\in I} \subseteq \mathbb{R}$ be a sequence of real numbers. We say that a is the **ultralimit**, denoted by $a := \lim_{\mathcal{F}} a_i$ if for every $\varepsilon > 0$ we have

$$\{i \in I : |a_i - a| < \varepsilon\} \in \mathcal{F}.$$

Define the measure for any decomposable set $X = \prod_{i\in I} A_i/\mathcal{F}$ as

$$\mu(X) = \lim_{\mathcal{F}} (\mu \times \nu)_i(A_i).$$

Current progress

The measure is invariant under translation by a constant

Let $(X, \mathcal{A}, \mu)$ be a measure space, and $(\mathbb{R}, \mathcal{B}, \nu)$ be a real measure space equipped with the Lebesgue measure.
Let $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ be a product measure space.
Then, for any $c \in \mathbb{R}$ and $E \in \mathcal{A} \otimes \mathcal{B}$

$$\mu \times \nu (E + c) = \mu \times \nu (E) .$$

- The statement may be proven via the ultraproduct construction in case where $(X, \mathcal{A}, \mu)$ is a σ-finite measure space.

The next step

Future work

The measure is invariant under translation by a constant

Let $(X, \mathcal{A}, \mu)$ be a measure space containing an atom of infinite measure, and $(\mathbb{R}, \mathcal{B}, \nu)$ be a real measure space equipped with the Lebesgue measure.

Let $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ be a product measure space.

Then, for any $c \in \mathbb{R}$ and $E \in \mathcal{A} \otimes \mathcal{B}$

$$\mu \times \nu (E + c) = \mu \times \nu (E) .$$

Thank you for your attention!

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