Gauge theory, knots and gravity

Directed studies report

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Abstract

In this semester, I have read the book "Gauge Fields, Knots and Gravity" by John Baez and Javier P. Muniain [1], which is mainly concerned about the formalism of quantum gravity. During this process I became familiar with concepts like deRham cohomology, Chern-Simons theory, Chern classes, knot theory, Palatini formalism, the Wheeler-deWitt equation and many more. The current report only gives an overview, hence it does not provide proof for every statement and avoids deeper, more detailed explanations.

1 Electromagnetism

1.1 deRham cohomology

The set of p-forms on a manifold M is denoted by $\Omega^p(M)$, and the exterior derivative is denoted by $d : \Omega^p(M) \to \Omega^{p+1}(M)$.

Definition 1.1 (Closed forms)**.** A form $\omega \in \Omega^p(M)$ is **closed**, when $d\omega = 0$. The set of closed forms is denoted as $Z^p(M)$.

Definition 1.2 (Exact forms)**.** A form $\omega \in \Omega^p(M)$ is **exact**, when $\exists \xi \in \Omega^{p-1}(M) : d\xi = \omega$. The set of exact forms is denoted as $B^p(M)$.

Proposition 1.0.1. *$Z^p(M)$ and $B^p(M)$ are both vector spaces.*

Proposition 1.0.2. *All exact forms are closed, but the opposite is not necessarily true, and hence $B^p(M) \subseteq Z^p(M)$. Moreover, $d^2 = 0$.*

Definition 1.3. The vector space

$$H^p(M) = B^p(M)/Z^p(M) \tag{1}$$

is called the **p-th deRham cohomology group of** M.

Definition 1.4 (Cohomologous forms)**.** Two forms are **cohomologous**, if they differ by an exact form.

1.2 Maxwell equations

Let M be a 4-dimensional manifold with a semi-Riemannian metric g. The Maxwell equations are

$$dF = 0, \tag{2}$$

$$\star d \star F = J, \tag{3}$$

where F is a 2-form, J is a 1-form, $\star : \Omega^k(M) \to \Omega^{4-k}(M)$ is the Hodge star operator. Note, that if F is exact, i.e. there is a one-form $A \in \Omega^1(M)$ (called vector potential) such that $F = dA$, $dF = 0$ is automatically satisfied. Choosing local coordinates x^0, x^1, x^2, x^3, one can split F as

$$F = B + E \wedge dx^0, \tag{4}$$

where $B \in \Omega^2(M)$ corresponds to the magnetic field and $E \in \Omega^1(M)$ corresponds to the electric field.

2 Gauge fields

2.1 Gauge groups and gauge transformations

Let $E \xrightarrow{\pi} M$ be a vector bundle. Let $\{U_\alpha\}$ be a cover of the base manifold M by open sets. By definition of a vector bundle the fibers of the bundle are isomorphic to a single vector space V. Let G a group and $\rho : G \to \mathrm{End}(V)$ be a representation of the group G. One can "glue" together trivial bundles of the form $U_\alpha \times V$ using **transition functions**

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G, \tag{5}$$

via the mapping

$$v_{(\alpha)} = \rho(g_{\alpha\beta})v_{(\beta)}, \tag{6}$$

where $(p, v_{(\alpha)}) \in U_\alpha \times V$ and $(p, v_{(\beta)}) \in U_\beta \times V$. This construction is called a G**-bundle** and V is called the **standard fiber**.

Definition 2.1 (Living in a group)**.** Let G be a group. A linear transformation $T : E_p \to E_p$ **lives in** G, if it is of the form

$$T : [p, v]_\alpha \mapsto [p, \rho(g)v]_\alpha, \tag{7}$$

for some $g \in G$.

Definition 2.2 (Living in a Lie algebra)**.** Let $\mathfrak{g}$ be a Lie algebra. A linear transformation $T : E_p \to E_p$ **lives in** $\mathfrak{g}$, if it is of the form

$$T : [p, v]_\alpha \mapsto [p, d\rho(g)v]_\alpha, \tag{8}$$

for some $g \in G$.

Remark. If $T \in \mathrm{End}(E)$ for a vector bundle E, we say that T lives in a Lie algebra $\mathfrak{g}$ if it lives in the same Lie algebra $\mathfrak{g}$ in each point.

Definition 2.3 (Gauge invariance)**.** Let $s \in \Gamma(M)$ be a section of a G-bundle $E \xrightarrow{\pi} M$. Suppose that s satisfies a differential equation. If $\rho(g)s$ is also a solution for all $g \in G$, then the differential equation is called **gauge invariant**.

Definition 2.4 (Holonomy)**.** Let $\lambda : [0, T] \to M$ be a smooth path from p to q on the manifold M. Let $E \xrightarrow{\pi} M$ be a vector bundle with connection D. Then the operator which parallel transports any vector $v \in E_p$ to E_q is called **holonomy**:

$$H(\lambda, D) : E_p \to E_q \tag{9}$$

Remark. The holonomy $H(\lambda, D)$ is a linear transformation.

Definition 2.5 (Wilson loop)**.** Let D be a connection on a bundle $E \xrightarrow{\pi} M$. Then the **Wilson loop** is defined as

$$W(\lambda, D) := \mathrm{tr}(H(\lambda, D)), \tag{10}$$

where H is the holonomy and $\lambda : [0, t] \to M$ is a loop, i.e. $\lambda(0) = \lambda(t)$.

Lemma 2.1. *The Wilson loop is gauge invariant.*

Proof. Let $\gamma : [0, T] \to M$ be a loop and let $u : \mathrm{im}\,\gamma \to E$ be a vector field over γ. Let D, D' be connections on the bundle $E \xrightarrow{\pi} M$ where D' differs from D by a gauge transformation $g \in \mathcal{G}$. The holonomy $H(\gamma, D)$ is sending $u(0)$ to $u(T)$, and the holonomy with the other connection $H(\gamma, D)$ is sending $g(\gamma(0))u(0)$ to $g(\gamma(T))u(T)$. We also know that the holonomy is linear, hence

$$H(\gamma, D') = g(\gamma(T))H(\gamma, D)g(\gamma(0))^{-1}. \tag{11}$$

Since γ is a loop, $g(\gamma(T)) = g(\gamma(0))$, therefore

$$\mathrm{tr}\, H(\gamma, D') = \mathrm{tr}\left(g(\gamma(0))H(\gamma, D)g(\gamma(0))^{-1}\right) = \mathrm{tr}\, H(\gamma, D), \tag{12}$$

by cyclicity of the trace. $\square$

Definition 2.6 (Curvature)**.** Let v, w be vector fields on a manifold M. The curvature tensor is a $\mathrm{End}(E)$-valued 2-form

$$F(v,w) = [D_v, D_w] - D_{[v,w]}. \tag{13}$$

Remark. A connection with vanishing curvature $F(v,w)s = 0$ for all vector fields v, w, s are called **flat**.

Definition 2.7 (Exterior covariant derivative)**.** Let $E \xrightarrow{\pi} M$ a vector bundle. The **exterior covariant derivative** is defined as

- $d_D s(v) = D_v s$ for s section on E,
- $d_D(s \otimes \omega) = D_v s \wedge \omega + s \otimes d\omega$ for s section on E and $\omega \in \Omega(M)$.

Theorem 2.2 (Bianchi identity)**.** *The Bianchi identity states that*

$$d_D F = 0, \tag{14}$$

where F is the curvature tensor and d_D is the exterior covariant derivative.

2.2 Yang-Mills theory

Definition 2.8 (Yang-Mills Lagrangian)**.** The **Yang-Mills Lagrangian** is defined as

$$\mathcal{L}_{YM} = \frac{1}{2}\,\mathrm{tr}(F \wedge \star F). \tag{15}$$

Definition 2.9 (Yang-Mills action)**.** The **Yang-Mills action** is defined as

$$S_{YM} = \frac{1}{2}\int_M \mathrm{tr}(F \wedge \star F). \tag{16}$$

Proposition 2.2.1. *The action principle $\delta S_{YM} = 0$ recovers the Yang-Mills equations.*

2.3 Chern classes

In contrast to the Yang-Mills action, we are trying to write down a metric-independent action. In 4 dimensions, it could look something like

$$S(A) = \int_M \mathrm{tr}(F \wedge F) \tag{17}$$

This motivates the following definition:

Definition 2.10 (Chern form)**.** Let M be a $2n$-dimensional manifold. Then the **Chern form** is defined as

$$\mathrm{tr}(F^n). \tag{18}$$

Proposition 2.2.2. *The action*

$$S(A) = \int_M \mathrm{tr}(F^n) \tag{19}$$

does not depend on the connection A, and it only depends on the bundle $E \xrightarrow{\pi} M$.

Proof. We show this by varying S. Let $n = \dim M$ and write

$$\begin{aligned}
\delta S &= \int_M \delta\,\mathrm{tr}(F^n) && (20)\\
&= n\int_M \delta\,\mathrm{tr}(\delta F \wedge F^{n-1}) && (21)\\
&= n\int_M \delta\,\mathrm{tr}(d_D \delta A \wedge F^{n-1}) && (22)\\
&= n\int_M \delta\,\mathrm{tr}(\delta A \wedge d_D F^{n-1}) = 0, && (23)
\end{aligned}$$

since $d_D F^{n-1} = 0$ by the Bianchi identity. $\square$

Proposition 2.2.3 (Chern forms are closed)**.**

Proof.

$$d\,\mathrm{tr}\,F^k = \mathrm{tr}(d_D F \wedge F^{k-1} \dots) = 0, \tag{24}$$

since $d_D F = 0$ by Theorem 2.2. □

Proposition 2.2.4 (Chern forms changes by an exact form)**.** *By varying the connection A, the Chern form changes by an exact form.*

Proof.

$$\delta\,\mathrm{tr}\,F^k = k\,\mathrm{tr}(\delta F \wedge F^{k-1}) = k\,\mathrm{tr}(d_D\delta A \wedge F^{k-1}) = kd\,\mathrm{tr}(\delta A \wedge F^{k-1}). \tag{25}$$

When integrating this and using Stokes' theorem, it can be easily shown that the difference of two Chern forms corresponding to different A vector potentials is exact. □

Definition 2.11 (Chern class)**.** Since the Chern forms are closed and their difference is exact by changing the vector potential, the Chern form defines a cohomology class in $H^{2k}(M)$, called the **Chern class**.

Remark. In a special case, we will construct the form whose exterior derivative is the Chern form, which is called the Chern-Simons form in Theorem 2.3, i.e. the Chern class is 0.

Proposition 2.2.5 (Integrality of the Chern class)**.** *If $E \xrightarrow{\pi} M$ is a complex vector bundle. Then*

$$\frac{i^k}{(2\pi)^k k!}\,\mathrm{tr}\,F^k \tag{26}$$

is an integral class, meaning that

$$\frac{i^k}{(2\pi)^k k!}\int_M \mathrm{tr}\,F^k \tag{27}$$

is an integer if M is compact and oriented.

Consider the case of electromagnetism, and let A be the vector potential on a $U(1)$-bundle E. When E is trivial, it can be shown that $\mathrm{tr}\,F = iB$, where B is the 2-form from electromagnetism corresponding to the magnetic field. For non-trivial bundles, by the integrality of the Chern class, we must have that

$$\int_{S^2} B = 2\pi N, \tag{28}$$

for some integer $N \in \mathbb{Z}$ and where B is now a generalization of the magnetic field for non-trivial bundles. This is interesting because this seemingly predicts magnetic monopoles to exist, though these have not yet been observed in Nature.

2.4 Chern-Simons theory

Theorem 2.3 (Chern-Simons form)**.** *Let $E \xrightarrow{\pi} M$ be a trival vector bundle with connection D. The Chern form $\mathrm{tr}(F \wedge F)$ is exact, and*

$$\mathrm{tr}(F \wedge F) = d\,\mathrm{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) \tag{29}$$

Proof. Let $A_s = sA$. Then let us denote $F_s := F(A_s) = d_D A_s = sdA + s^2 A \wedge A$ the curvature of the connection A_s. We also know that $\frac{dF_s}{ds} = \frac{dd_D A_s}{ds} = \frac{d(sd_D A)}{ds} = d_D A$. Then we can calculate

$$\mathrm{tr}(F \wedge F) = \int_0^1 \frac{d}{ds}\,\mathrm{tr}(F_s \wedge F_s) ds = 2\int_0^1 \mathrm{tr}\left(\frac{dF_s}{ds} \wedge F_s\right) ds = 2d\int_0^1 \mathrm{tr}\left(A \wedge F_s\right) ds \tag{30}$$

$$= 2d\int_0^1 \mathrm{tr}\left(sA \wedge dA + s^2 A \wedge A \wedge A\right) ds = d\,\mathrm{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) \tag{31}$$

□

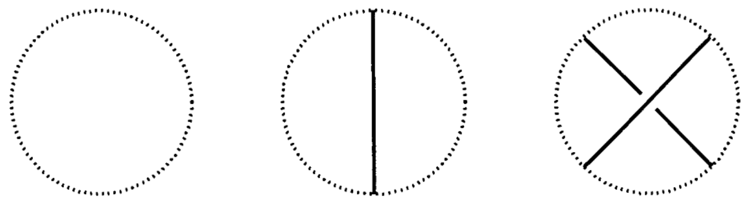

Figure 1: Local scenes of the link diagrams. The dashed line represents a boundary of a small disk on the plane where the link is projected.

Definition 2.12 (Chern-Simons action)**.** Let $M \cong \mathbb{R} \times S$. The Chern-Simons action is defined as

$$S_{CS}(A) = \int_S \operatorname{tr}\left(A \wedge d_S A + \frac{2}{3} A \wedge A \wedge A\right) \tag{32}$$

This action is invariant under **small gauge transformations**, i.e. gauge transformations connected to the identity $e \in G$. Under **large gauge transformations**, it changes by an integer multiple of $8\pi^2$, which is the consequence of the integrality of the second Chern class.

2.5 Link invariants

Definition 2.13 (Knot)**.** A **knot** is a submanifold of $\mathbb{R}^3$ diffeomorphic to S^1.

Definition 2.14 (Link)**.** A **link** L is a submanifold of $\mathbb{R}^3$ which is diffeomorphic to disjoint union of circles.

Definition 2.15. Two links are said to be **ambient isotopic**, if there is a smooth map

$$\alpha : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^3, \tag{33}$$

such that

- $\forall t \in [0, 1] : \alpha_t := \alpha(t, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism,
- α_0 is the identity,
- α_1 maps L to L'.

In order to investigate links, one usually investigates 2-dimensional projections of the links. On these diagrams the overlapping of crossings are also denoted. The diagram should consist of certain so-called "local scenes" to properly encode the information encoded in the link, demonstrated by Figure 1.

Theorem 2.4. *In local frames, ambient isotopy corresponds to so-called Reidemeister moves given by*

- *0: Deformation,*
- *I: Twisting and untwisting a string,*
- *II: Moving a string under another string, creating 2 crossings,*
- *III: Moving a string completely over or under a crossing.*

This is demonstrated by Figure 2.

In order to classify knots, we need to find quantities which are *invariant* under ambient isotopy. Invariance under ambient isotopy is equivalent to invariance under Reidemeister moves. This motivates the following definition:

Definition 2.16 (Link invariant)**.** The link invariant is a function $f : L \to ?$ which is invariant under ambient isotopy, i.e. given any ambient isotopy $\alpha : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^3$ we have

$$f(\alpha(t, L)) = f(L) \qquad \forall t \in [0, 1]. \tag{34}$$

There are some invariants, which are not invariant under the Reidemeister moves, but invariant under the so-called framed Reidemeister moves. To define these, we need to define framed links.

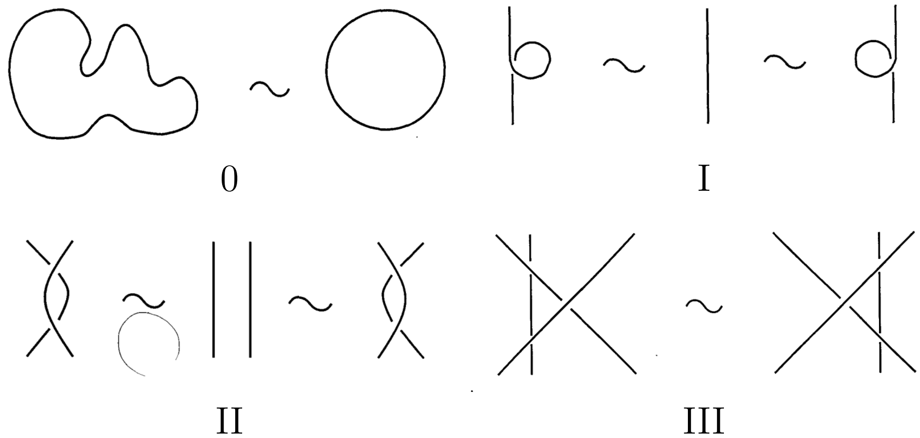

Figure 2: Reidemeister moves.

Figure 3: Modified first Reidemeister move I'.

Definition 2.17 (Framed link)**.** A framing on a link L is a vector field $v_p \in T_p\mathbb{R}^3$ on $p \in L$ such that $\forall p \in L : v_p \notin T_pL$.

Definition 2.18 (Ambient isotopy of framed links)**.** Two framed links are isotopic if there is an ambient isotopy taking the first link to the second link which also takes the framing from the first link to the second link's framing.

In terms of framed links, the Reidemeister move I no longer corresponds to ambient isotopy, and one needs to introduce a framed Reidemeister move labeled as I', seen in Figure 3. Hence, the framed Reidemeister links are 0, I', II, III.

For defining several link invariants, we need to differentiate left-handed and right-handed crossings of oriented links.

Definition 2.19 (Sign of a crossing)**.** For an oriented link, we assign $+1$ to a crossing if it is right-handed, and -1 if it is left handed, and it will be denoted as $\text{sign}(p)$ for a crossing p of L, and it is demonstrated by Figure 4.

Definition 2.20 (Writhe)**.** Let L be a link. The writhe is simply equal to the sum of all signs of the crossing.

$$w(L) = \sum_{\text{crossing } p} \text{sign}(p) \tag{35}$$

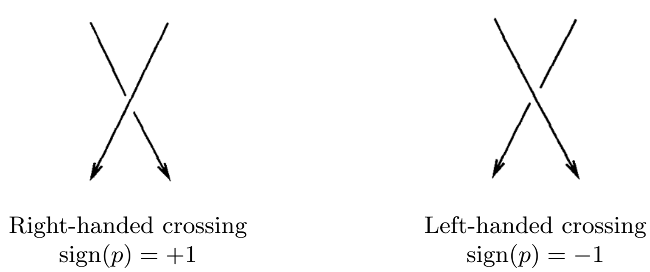

Figure 4: Signs corresponding to crossings.

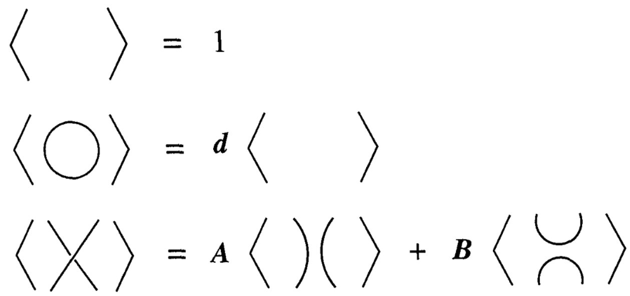

Figure 5: Skein relations for defining the Kauffman bracket.

Proposition 2.4.1. *The writhe is invariant under the framed Reidemeister moves.*

Definition 2.21 (Kauffman bracket)**.** The Kauffman bracket is a polynomial in variables A, B, d and written as

$$\langle L \rangle =: \langle L \rangle(A, B, d) := \sum_{\text{states } \sigma} d^{||\sigma||} \prod_{\text{crossings } p} \sigma(p), \tag{36}$$

where σ is called the **state** and it is a possible final descendant after splicing the link with the Kauffman bracket skein relations until we get to unknots as seen in Figure 5 The final number of unknots is $||\sigma||$ and is called the **loop value**.

Theorem 2.5. *The Kauffman bracket is invariant under the framed Reidemeister moves if and only if*

$$B = A^{-1}, \tag{37}$$
$$d = -(A^2 + A^{-2}), \tag{38}$$

and by this choice the Kauffman bracket is also redefined to be a Laurent polynomial in A, and written as $\langle L \rangle(A)$ for any link L.

Using the Kauffman bracket, the Jones polynomial is relatively easy to define:

Definition 2.22 (Jones polynomial)**.** The Jones polynomial for any link L is defined by the Laurent polynomial

$$V_L(A) = (-A^{-3})^{w(L)} \langle L \rangle(A), \tag{39}$$

where $w(L)$ is the writhe of L.

To uncover the connection between the Kauffman bracket and Chern-Simons theory, we need to define vacuum expectation values, which correspond to expectation values of physical observable quantities.

Definition 2.23 (Vacuum expectation value)**.** Let $\mathcal{A}$ be the space of all G-connections, where G is the gauge group. Let us consider a function $f : \mathcal{A} \to \mathbb{R}$, which is called an observable. The **vacuum expectation value** of this observable is

$$\langle f \rangle = \frac{1}{Z} \int_{\mathcal{A}} f(A) e^{\frac{1}{\hbar} S_{YM}(A)} \mathcal{D}A, \tag{40}$$

where

$$Z = \int_{\mathcal{A}} e^{\frac{1}{\hbar} S_{YM}(A)} \mathcal{D}A \tag{41}$$

is called the **partition function**. [1]

$$\mathcal{L}(L) = \int_{\mathcal{A}} W(\hat{\gamma}_1, A) \dots W(\hat{\gamma}_n, A) e^{\frac{ik}{4\pi} S_{CS}(A)} \mathcal{D}A \tag{42}$$

and for the fundamental representation of $U(1)$ one obtains

$$\mathcal{L}(L) = e^{i \frac{\pi}{k} \omega(L)}, \tag{43}$$

[1]The term $\mathcal{D}A$ does not necessarily correspond to a well-defined measure. However, in theoretical physics these kinds of "path integrals" are frequently used.

where $\omega(L)$ is the writhe of the link. When one considers the fundamental representation of $SU(2)$, one obtains

$$\mathcal{L}(L) = \langle L \rangle, \tag{44}$$

where $A = q^{\frac{1}{4}}$ and $q = e^{i\frac{2\pi}{k+2}}$.

3 Gravity

3.1 Einstein-Hilbert action

Definition 3.1 (Riemann curvature tensor)**.** Let u, v be vector fields on M. Then

$$R(u, v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]} \tag{45}$$

.

The components of the Riemann curvature tensor can be written by

$$R^i{}_{jkl} = g(\partial_i, R(\partial_j, \partial_k)\partial_l). \tag{46}$$

Definition 3.2 (Ricci curvature tensor)**.** The Ricci curvature tensor is defined[2] as

$$R_{ij} := R^k{}_{ikj}. \tag{47}$$

Definition 3.3 (Ricci scalar)**.** The Ricci scalar defined by

$$R := g^{ij} R_{ij}, \tag{48}$$

where $g^{ij} := (g^{-1})_{ij}$ denotes the components of the inverse metric tensor.

Definition 3.4. The **Einstein-Hilbert** action S is defined as

$$S(g) = \int_M R \operatorname{vol}. \tag{49}$$

Proposition 3.0.1. *If the variation on the metric tensor g vanishes outside a compact set, the following implication is true:*

$$\delta S = 0 \implies R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 0, \tag{50}$$

which is called Einstein's equations.

3.2 Palatini formalism

We aim to rewrite the Einstein-Hilbert action

$$S(g) = \int_M R \operatorname{vol}, \tag{51}$$

so that it is not a function of the metric, but a function of the connection and a "frame field".

Definition 3.5 (Frame field)**.** A **frame field** of the tangent bundle TM is a vector bundle isomorphism (or trivialization)

$$e : M \times \mathbb{R}^n \to TM \tag{52}$$

sending each fiber $\{p\} \times \mathbb{R}^n$ (where $p \in M$) to the corresponding tangent space T_pM.

Remark. If $\dim M = 4$, e is also called a **tetrad**.

Remark. Since the frame field is a vector bundle isomorphism by definition, it has an inverse $e^{-1} : TM \to M \times \mathbb{R}^n$.

[2]For repated upper and lower indices, the Einstein summation convention is used.

Lemma 3.1 (Basis of sections)**.** *Given a section $s : M \to \mathbb{R}^n$, there is a natural* ***basis of sections*** $\xi_1, \ldots, \xi_n : M \to \mathbb{R}^n$ *defined by*

$$\xi_1(p) = (1, 0, \ldots, 0, 0), \tag{53}$$
$$\xi_2(p) = (0, 1, \ldots, 0, 0), \tag{54}$$
$$\vdots \tag{55}$$
$$\xi_n(p) = (0, 0, \ldots, 0, 1), \tag{56}$$
$$\tag{57}$$

so that we can write the section s as

$$s = s^I \xi_I \tag{58}$$

Remark. Here $\mathbb{R}^n$ is called the **internal space**, and we denote **internal indices** by latin upper-case letters starting from I such as I, J, K, L.

Definition 3.6 (Canonical inner product on local sections)**.** Given two sections s, s' on the bundle $M \times \mathbb{R}^n$, we define the **canonical inner product** as

$$\eta : M \times \mathbb{R}^n \times M \times \mathbb{R}^n \to \mathbb{R} \tag{59}$$
$$\eta(s, s') = \eta_{IJ} s^I s'^J, \tag{60}$$

where $\eta_{IJ} \in \mathbb{R}^{n \times n}$ is the Minkowski metric.

Definition 3.7 (Orthonormal frame field)**.** A frame field $e : M \times \mathbb{R}^n \to TM$ is orthonormal, if

$$g(e(\xi_I), e(\xi_J)) = \eta_{IJ} \tag{61}$$

Proposition 3.1.1. *For any s, s' sections on the bundle $M \times \mathbb{R}^n$ and orthonormal frame field $e : M \times \mathbb{R}^n \to TM$ we have that*

$$g(e(s), e(s')) = \eta(s, s'). \tag{62}$$

Definition 3.8 (Lorentz connection)**.** Let s, s' sections on the bundle $M \times \mathbb{R}^n$ and v vector field on M. A Lorentz connection is any connection which satisfies

$$v\eta(s, s') = \eta(D_v s, s') + \eta(s, D_v s'). \tag{63}$$

Let us transfer a Lorentz connection D_v from the trivial bundle $M \times \mathbb{R}^n$ to the tangent bundle TM via the frame field e. Then we get an "imitation connection" $\tilde{\nabla}$. With this, we can define the "imitation Riemann curvature tensor" as the following:

Definition 3.9 (Imitation Riemann curvature tensor)**.**

$$\tilde{R}^{\gamma}{}_{\alpha\beta}{}^{\delta} = F^{IJ}_{\alpha\beta} e^{\gamma}_I e^{\delta}_J \tag{64}$$

Definition 3.10 (Palatini action)**.** The **Palatini action** is given by

$$S(A, e) = \int_M e^{\alpha}_I e^{\beta}_j F^{IJ}_{\alpha\beta} \,\mathrm{vol} \tag{65}$$

Proposition 3.1.2. *Varying the frame field e in the Palatini action we get that*

$$\delta_e S = 0 \implies \tilde{R}_{\alpha\beta} - \frac{1}{2}\tilde{R} g_{\alpha\beta} = 0. \tag{66}$$

Proposition 3.1.3. *Varying the Lorentz connection D in the Palatini action we get that*

$$\delta_D S = 0 \implies \tilde{\nabla} = \nabla. \tag{67}$$

3.3 ADM formalism

Consider the case of Lorentzian manifolds which are diffeomorphic to $\mathbb{R} \times S$, where S is a 3-dimensional manifold representing space and $t \in \mathbb{R}$ represents time, i.e.

$$\phi : M \to \mathbb{R} \times S. \tag{68}$$

We could define a **time coordinate** on M by pulling back t, i.e.

$$\tau = \phi^* t. \tag{69}$$

$\Sigma \subset M$ is called a **slice** of M if it equals $\{\tau = \text{constant}\}$. There are two choices for the normal vector field n for which $n \perp \Sigma$, and from now on assume that this choice is fixed. One can derive formulas relating the manifold M and the submanifold Σ, which are called the **Gauss-Codazzi equations**:

$$R(\partial_i, \partial_j)\partial_k = ({}^3\nabla_i K_{jk} - {}^3\nabla_j K_{ik})n + ({}^3R^m_{ijk} + K_{jk}K^m_i - K_{ik}K^m_j)\partial_m, \tag{70}$$

where ${}^3\nabla$ is the intrinsic connection on Σ, ${}^3R$ is the Riemann curvature tensor on Σ and $K_{ij} := -g(\nabla_i \partial_j, n)$ is the extrinsic curvature. This means that the Gauss-Codazzi equation is a constraint on the metric.

Now we aim to rewrite the Einstein-Hilbert Lagrangian in terms of the intrinsic metric tensor ${}^3g_{ij}$. From now on, denote ${}^3g_{ij} = q_{ij}$. We can simply split ∂_τ as

$$\partial_\tau = -g(\partial_\tau, n)n + (\partial_\tau + g(\partial_\tau, n)n) =: Nn + \vec{N}, \tag{71}$$

where $n \perp \Sigma$, and where $N := -g(\partial_\tau, n)$ is called the **lapse function** and $\vec{N} = \partial_\tau + g(\partial_\tau, n)n$ is the **shift vector**. The Einstein-Hilbert Lagrangian density is given by

$$\mathcal{L} = \det(q)^{\frac{1}{2}} N({}^3R + \operatorname{tr}(K^2) - \operatorname{tr}(K)^2). \tag{72}$$

The conjugate momentum is defined by

$$p^{ij} = \frac{\partial \mathcal{L}}{\partial q_{ij}} \tag{73}$$

which we can use to define the Hamiltonian density by

$$\mathcal{H} = p_{ij}\dot{q}^i j - \mathcal{L} = \det(q)^{\frac{1}{2}}(NC - N^i C_i), \tag{74}$$

where

$$C = -{}^3R + \det(q)^{-1}\left(\operatorname{tr}(p^2) - \frac{1}{2}\operatorname{tr}(p)^2\right) = -2G_{\mu\nu}n^\mu n^\nu \tag{75}$$

$$C_i = -2{}^3\nabla^j(\det(q)^{-\frac{1}{2}}p_{ij}) = -2G_{\mu i}n^\mu. \tag{76}$$

This implies that for the Hamiltonian we get

$$H = \int_\Sigma \mathcal{H} d^3x = 0, \tag{77}$$

which sets a constraint on the metric tensor.

3.4 Wheeler-deWitt equation

Now we want to find a Hilbert space for the theory on which we could find counterparts to q_{ij} and p^{ij} which are operators on this Hilbert space, and denoted by $\hat{q}_{ij}$ and $\hat{p}^{ij}$. This is called quantization, and it is a common procedure in quantum theory in order to find predictions for quantum mechanical phenomena by classical mechanical analogy. This is highly non-obvious, and the book even states that it is not even clear, what the Hilbert space should be, and just pretends that we have a Hilbert space. Undeterred, let us denote $\mathrm{Met}(\Sigma)$ the set of all metric tensor fields on Σ, let the Hilbert space $\mathcal{H} := L^2(\mathrm{Met}(\Sigma))$ be the square-integrable functions on $\mathrm{Met}(\Sigma)$ (even if it is not well-defined) and let $\psi \in \mathcal{H}$ and suppose

$$\hat{q}_{ij}(x)\psi(q) = g_{ij}(x)\psi(q), \tag{78}$$

$$\hat{p}_{ij}(x)\psi(q) = -i\frac{\partial}{\partial q_{ij}(x)}\psi(q) \tag{79}$$

where $g_{ij} \in \mathrm{Met}(\Sigma)$, $x \in \Sigma$, and $\frac{\partial}{\partial q_{ij}(x)}$ is a functional derivative. With these operators, we can quantize the Hamiltonian, i.e. exchange all the q_{ij}, p^{ij} variables to $\hat{q}_{ij}$ and $\hat{p}^{ij}$. It is not clear at all, why this should work. The functions q_{ij}, p^{ij} are obviously commuting, while $\hat{q}_{ij}$ and $\hat{p}^{ij}$ do not commute. This yields operator ordering ambiguities, but we will ignore dealing with these now. By quantizing the Hamiltonian with a certain operator ordering, we acquire the **Wheeler-DeWitt** equation:

$$\hat{H}\psi = 0, \tag{80}$$

where $\hat{H} = \int_\Sigma N\hat{C} + N^i\hat{C}_i \det(q)^{\frac{1}{2}} d^3x$ and $\hat{C}, \hat{C}_i$ are the quantized versions of C, C_i. There are several problems regarding this equation

- In this form, no solution has been found to this equation.
- The Hilbert space is not well defined, and it is not clear at all that one has a well-defined inner product. This is called the **inner product problem**.
- Any operator commutes with the quantized Hamiltonian $\mathcal{H}$, which yields dynamics that is invariant under time translation. This is called the **problem of time**.

3.5 Relation between Chern-Simons theory and the Wheeler-DeWitt equation

One can show that Chern-Simons theory is intimately related to the Wheeler-DeWitt equation in a sense that the Chern-Simons action can be used to obtain a solution for the Wheeler-DeWitt equation. We begin by complexifying the theory and introducing new variables, the so-called Ashtekar variables.

Definition 3.11 (Complexified tangent bundle)**.** The complexified tangent bundle $\mathbb{C}TM \to M$ is a vector bundle whose fiber at each point $p \in M$ is $\mathbb{C} \otimes T_pM$.

As in the Palatini formalism, there is a complexified imitation tangent bundle $M \times \mathbb{C}^4$ and a complex frame field $e : M \times \mathbb{C}^4 \to \mathbb{C}TM$. Moreover, we can define an internal metric $\eta : M \times \mathbb{C}^4 \times M \times \mathbb{C}^4 \to \mathbb{C}$. Furthermore, this construction allows us to define a connection on the imitation tangent bundle as a $\mathrm{End}(\mathbb{C}^4)$-valued 1-form. Similarly to the Palatini formalism, we can define a Lorentz connection

Definition 3.12 (Lorentz connection)**.** A Lorentz connection A is a $\mathrm{End}(\mathbb{C}^4)$-valued 1-form if

$$A_\alpha^{IJ} = -A_\alpha^{JI}. \tag{81}$$

Due to this property, we can think about a Lorentz connection as a $\Lambda^2\mathbb{C}^4$-valued 1-form.

Analoguously to the Hodge star operator in the tangent bundle, one can define an internal Hodge star operator on the imitation tangent bundle as

$$*T^{IJ} = \frac{1}{2}\varepsilon^{IJ}_{KL}T^{KL}. \tag{82}$$

Using the internal Hodge star operator one can decompose a Lorentz connection A into self-dual (${}^+A$) and anti-self-dual (${}^-A$) parts as

$$A = {}^+A + {}^-A. \tag{83}$$

Proposition 3.1.4. *The curvature of a self-dual Lorentz-connection is self-dual.*

By reformulating general relativity using these variables, one can rewrite the Einstein-Hilbert Lagrangian to depend on the self-dual connection A and the complexified frame field, similarly to the Palatini formalism.

Definition 3.13 (Self-dual action)**.** The self-dual action is defined as

$$S_{SD}({}^+A, e) = \int_M e_I^\alpha e_J^\beta {}^+F_{\alpha\beta}^{IJ} \,\mathrm{vol} \tag{84}$$

Let us now work on the spacetime manifold M with a spacelike slice Σ, similarly to the ADM formalism. A self-dual connection ${}^+A_\alpha^{IJ}$ on M can be restricted to Σ, where it is again self-dual, and is denoted by A_i^{IJ}.[3] Continuing this, one eventually acquires a theory where the self-dual connections form the configuration space of the theory (i.e. it corresponds to the "position" in the Lagrangian formalism) and its momentum conjugate is related to the complexified frame field. Using these new Ashtekar variables, one can discover a surprising connection between Chern-Simons theory and quantum gravity.

[3]The plus sign is deliberately dropped here, it is customary.

Theorem 3.2. *Let us define the Chern-Simons state as*

$$\Psi_{CS}(A) = e^{-\frac{6}{\Lambda} S_{CS}(A)}, \tag{85}$$

where A is the connection on the manifold Σ and $\Lambda \in \mathbb{R}\backslash\{0\}$ is called the cosmological constant. The Chern-Simons state satisfies the equation

$$\hat{C}_j \Psi_{CS}(A) = \hat{C}\Psi_{CS}(A) = 0. \tag{86}$$

Remark. The "measure" $\Psi_{CS}(A)\mathcal{D}A$ corresponds to the Kauffman bracket link invariant as seen in Section 2.5.

References

[1] J. Baez and J. P. Muniain. *Gauge fields, knots and gravity.* 1995.