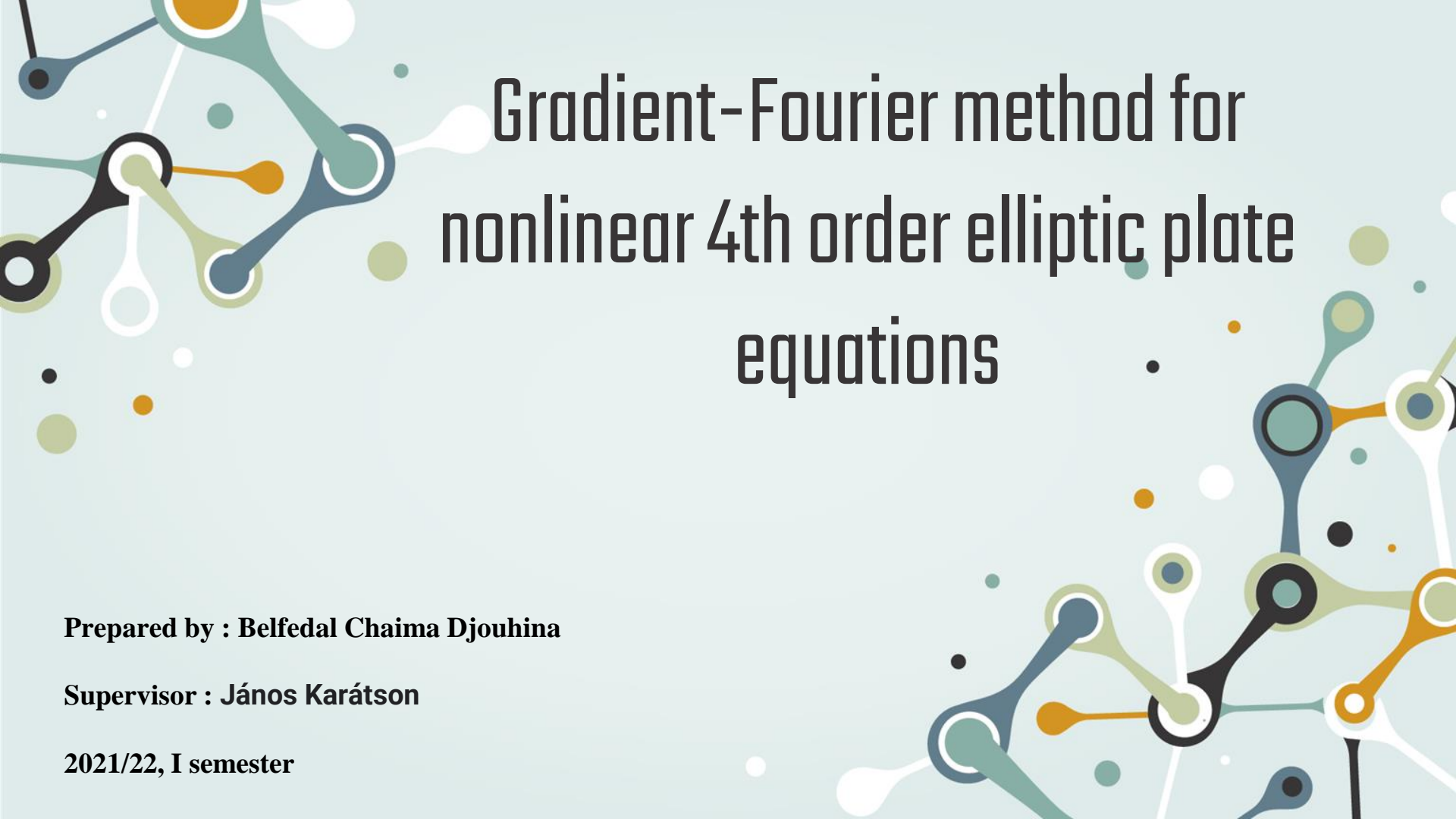


An abstract graphic design featuring organic, flowing shapes in shades of teal, orange, and black. A central orange circle contains the number '02' in white. To its left, a larger orange shape is partially enclosed by a black shape. Various other shapes, including circles and teardrop forms, are scattered around the central elements. The background is a light, pale blue.

02

Directed Studies 2



Gradient-Fourier method for nonlinear 4th order elliptic plate equations

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PURPOSE OF PROJECT



- Apply Gradient-Fourier method to the nonlinear plate equation (a 4th order PDE)



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1. The PDE model :

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain

$\langle f, g \rangle_{L^2(\Omega)} := \int_{\Omega} fg$ ($f, g \in L^2(\Omega)$) and let $H := (L^2(\Omega), \langle \cdot, \cdot \rangle_{L^2(\Omega)})$

We define a differential operator T with domain

$$\text{dom } T := D := \left\{ u \in H^4(\Omega) : u|_{\partial\Omega} = \frac{\partial^2 u}{\partial \nu^2} |_{\partial\Omega} = 0 \right\}$$

$$T(u) := \text{div}^2(\bar{g}(E(D^2u))\tilde{D}^2u) \quad (u \in D)$$

$$\bar{g}(E(D^2u)) = \left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 + \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2$$

$$\tilde{D}^2u = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \frac{\partial^2 u}{\partial y^2} & \frac{1}{2} \frac{\partial^2 u}{\partial x \partial y} \\ \frac{1}{2} \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \end{pmatrix}$$

Let \bar{g} be a real function that is C^2 in the variable r , and there exists $M, m, \lambda > 0$ such that

$$0 < m \leq \bar{g}(r) \leq M \quad (r \geq 0)$$

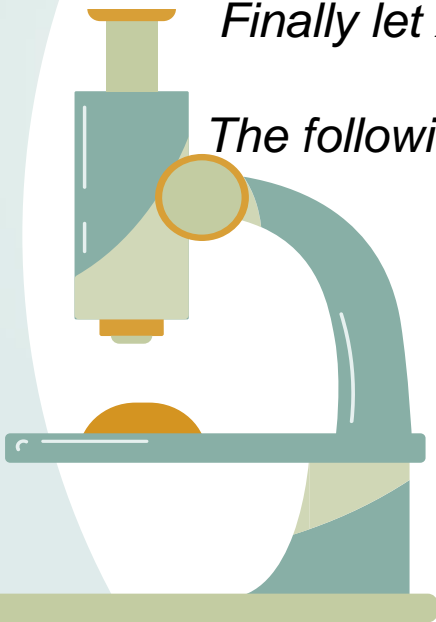
$$0 < m \leq (\bar{g}(r^2)r)' \leq M \quad (r \geq 0)$$

$$\left| \frac{\partial^2}{\partial r^2} (\bar{g}(r^2)r) \right| \leq \lambda \quad (r \geq 0)$$

Finally let $B := \Delta^2, \text{dom } B := D$

The following boundary value problem will be considered:

$$\begin{cases} T(u) = \alpha \\ u|_{\partial\Omega} = \frac{\partial^2 u}{\partial \nu^2}|_{\partial\Omega} = 0 \end{cases} \quad (1)$$



Theorem 1 : The unique weak solution $u^* \in H^2(\Omega) \cap H_0^1(\Omega)$ of the problem (1) :

$$\frac{1}{2} \int_{\Omega} \bar{g}(E(D^2 u^*)) (D^2 u^* \cdot D^2 v + \Delta u^* \Delta v) = \int_{\Omega} \alpha v \quad (v \in H_0^2(\Omega))$$

The operator F , defined by $\langle F(u), v \rangle$ on the left hand side satisfies

$$m \|h\|_{H_0^2(\Omega)}^2 \leq \langle F'(u)h, h \rangle_{H_0^2(\Omega)} \leq M \|h\|_{H_0^2(\Omega)}^2$$

Theorem 2 : Let $u_0 \in D$ be arbitrary .Then the following sequence:

$$u_{n+1} := u_n - \frac{2}{M+m} z_n \quad (n \in \mathbb{N})$$

where

$$\begin{cases} \Delta^2 z_n = T(u_n) - \alpha \\ z|_{\partial\Omega} = \frac{\partial^2 z}{\partial \nu^2} |_{\partial\Omega} = 0 \end{cases} \quad (2)$$

converges to the solution u^ and*

$$\|u^* - u^h\|_{H_0^2(\Omega)} \leq \frac{1}{m\sqrt{\lambda_1}} \|T(u_0) - g\|_{L^2(\Omega)} \left(\frac{M-m}{M+m}\right)^n,$$

where $\lambda_1 > 0$ is the smallest eigenvalue of (Δ^2) on D .

Remark:

(a) Weak form of the (2) :

$$\int_{\Omega} D^2 z_n D^2 v = \frac{1}{2} \int_{\Omega} \bar{g}(E(D^2 u))(D^2 u \cdot D^2 v + \Delta u \Delta v) - \int_{\Omega} \alpha v \quad (v \in H_0^2(\Omega))$$

(b) From now $\alpha > 0$ is constant .



The Gradient-Fourier method :

Let $\Omega \in [0, \pi]^2$, $\lambda_{k,l}$ and $e_{k,l}(k, l = 1, 2, \dots)$ denote the eigenvalues and eigenfunctions of (Δ^2) on D , respectively :

$$\lambda_{k,l} = (k^2 + l^2)^2, e_{k,l}(x, y) = \frac{2}{\pi} \sin(kx) \sin(ly).$$

Let us first introduce some notations. For $n=0, 1, \dots$, let

$$u_0 = 0$$
$$u_{n+1} := u_n - \frac{2}{M + m} z_n \quad (n \in \mathbb{N})$$



The Fourier method for the auxiliary equations:

Now let us focus on a single iteration step (i.e. $n \in \mathbb{N}$ is fixed in the section), where (2) is replaced by

$$\begin{cases} \Delta^2 z_n = r_n \\ z_n|_{\partial\Omega} = \frac{\partial^2 z_n}{\partial \nu^2} |_{\partial\Omega} = 0 \end{cases} \quad (3)$$

Let $c_{k,l}$ ($k, l = 1, 2, \dots$) be the coefficients of r_n in its Fourier series

expansion, that is $c_{k,l} := \int_{\Omega} r_n e_{k,l}$

Let N be a positive fixed integer and

$$r_n := \sum_{k,l=1}^N c_{k,l} e_{k,l}$$

Define z_n as
$$z_n := \sum_{k,l=1}^N \frac{c_{k,l}}{\lambda_{k,l}} e_{k,l}$$

A simple calculation shows that these satisfy (3) :

$$\Delta^2 z_n := \sum_{k,l=1}^N \frac{c_{k,l}}{\lambda_{k,l}} (\Delta^2 e_{k,l}) = \sum_{k,l=1}^N \frac{c_{k,l}}{\lambda_{k,l}} \lambda_{k,l} e_{k,l} = \sum_{k,l=1}^N c_{k,l} e_{k,l} = r_n$$

Calculation of the coefficients $C_{k,l}$:

$$\begin{aligned}c_{k,l} &= \int_{\Omega} (T(u) - \alpha) e_{k,l} \\ &= \int_{\Omega} \operatorname{div}^2 \bar{g}(E(D^2 u))(\tilde{D}^2 u) e_{k,l} - \int_{\Omega} \alpha e_{k,l} \\ &= \int_{\Omega} \bar{g} \left(\left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 + \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 \right) (\tilde{D}^2 u \cdot \tilde{D}^2 e_{k,l}) - \int_{\Omega} \alpha e_{k,l}\end{aligned}$$



$$J = \frac{\partial^2 u}{\partial x^2} = \sum_{k,l=1}^N d_{k,l} \frac{\partial^2 e_{k,l}}{\partial x^2} = - \sum_{k,l=1}^N \frac{2}{\pi} d_{k,l} k^2 \sin(kx) \sin(ly)$$

$$E = \frac{\partial^2 u}{\partial y^2} = \sum_{k,l=1}^N d_{k,l} \frac{\partial^2 e_{k,l}}{\partial y^2} = - \sum_{k,l=1}^N \frac{2}{\pi} d_{k,l} l^2 \sin(kx) \sin(ly)$$

$$G = \frac{\partial^2 u}{\partial x \partial y} = \sum_{k,l=1}^N d_{k,l} \frac{\partial^2 e_{k,l}}{\partial x \partial y} = \sum_{k,l=1}^N \frac{2}{\pi} d_{k,l} kl \cos(kx) \cos(ly)$$

$$Q = \frac{\partial^2 e_{k,l}}{\partial x^2} = -\frac{2}{\pi} k^2 \sin(kx) \sin(ly)$$

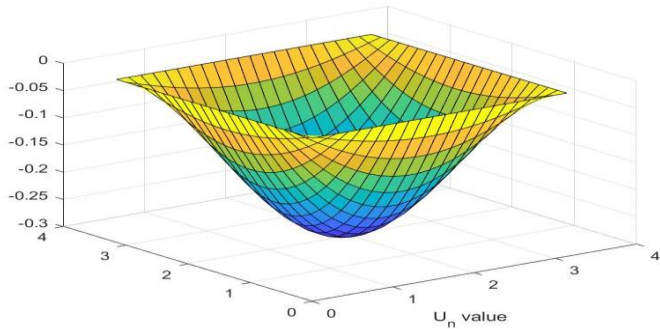
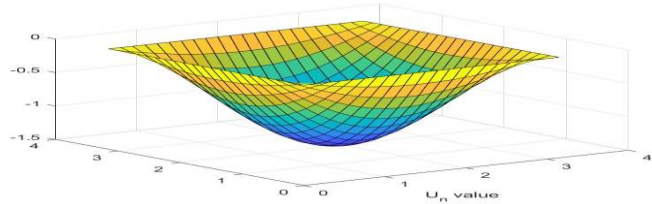
$$M = \frac{\partial^2 e_{k,l}}{\partial y^2} = -\frac{2}{\pi} l^2 \sin(kx) \sin(ly)$$

$$K = \frac{\partial^2 e_{k,l}}{\partial x \partial y} = \frac{2}{\pi} k * l \cos(kx) \cos(ly)$$

$$c_{k,l} = \int_{\Omega} \bar{g}((J)^2 + J * E + (E)^2 + (G)^2) * \left(\left(J + \frac{1}{2}E \right) * Q + G * \right)$$

Numerical experiment :

I have code it in Matlab program



```
>> gmiter4min(22,3)
```

```
errz =
```

```
15.0368
```

```
**
```

```
errz =
```

```
13.1337|
```

```
errz =
```

```
11.7817
```


THANK YOU

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