02

Directed Studies 2

Gradient-Fourier method for nonlinear 4th order elliptic plate equations

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PURPOSE OF PROJECT

- ➔ Apply Gradient-Fourier method to the nonlinear plate equation (a 4th order PDE)

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1.The PDE model :

Let $\Omega \subset \mathbb{R}^N$ *be a bounded domain*

$\langle f, g \rangle_{L^2(\Omega)} := \int_\Omega fg \ \ (f, g \in L^2(\Omega)$ *and let* $H := \left(L^2(\Omega), \langle .,. \rangle_{L^2(\Omega)}\right)$

We define a differential operator T with domain

$$dom\, T := D := \left\{ u \in H^4(\Omega) : u_{|\partial\Omega} = \frac{\partial^2 u}{\partial v^2}|_{\partial\Omega} = 0 \right\}$$

$$T(u) := div^2\left(\bar{g}\left(E(D^2u)\right)\widetilde{D}^2 u\right) \quad (u \in D)$$

$$\bar{g}\left(E(D^2u)\right) = \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2$$

$$\widetilde{D}^2 u = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u}{\partial y^2} & \frac{1}{2}\frac{\partial^2 u}{\partial x \partial y} \\ \frac{1}{2}\frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \end{pmatrix}$$

Let $\overline{g}$ be a real function that is C^2 in the variable r, and there exists M, $m, \lambda > 0$ such that

$$0 < m \leq \overline{g}(r) \leq M \qquad (r \geq 0)$$

$$0 < m \leq (\overline{g}(r^2)r)' \leq M \qquad (r \geq 0)$$

$$\left|\frac{\partial^2}{\partial r^2}(\overline{g}(r^2)r\right| \leq \lambda \qquad (r \geq 0)$$

Finally let $B := \Delta^2, dom\ B := D$

The following boundary value problem will be considered:

$$\begin{cases} T(u) = \alpha \\ u_{|\partial\Omega} = \frac{\partial^2 u}{\partial v^2}|_{\partial\Omega} = 0 \end{cases} \qquad (1)$$

<u>Theorem 1 :</u> The unique weak solution $u^* \in H^2(\Omega) \cap H_0^1(\Omega)$ of the problem (1) :

$$\frac{1}{2}\int_\Omega \overline{g}(E(D^2u^*))(D^2u^*.D^2v + \Delta u^*\Delta v) = \int_\Omega \alpha v \qquad \left(v \in H_0^2(\Omega)\right)$$

The operator F, defined by $\langle F(u), v\rangle$ on the left hand side satisfies

$$m\|h\|^2_{H_0^2(\Omega)} \leq \langle F'(u)h, h\rangle_{H_0^2(\Omega)} \leq M\|h\|^2_{H_0^2(\Omega)}$$

Theorem 2 : Let $u_0 \in D$ be arbitrary .Then the following sequence:

$$u_{n+1} := u_n - \frac{2}{M+m} z_n \quad (n \in \mathbb{N})$$

where

$$\begin{cases} \Delta^2 z_n = T(u_n) - \alpha \\ z_{|\partial\Omega} = \frac{\partial^2 z}{\partial v^2}|_{\partial\Omega} = 0 \end{cases} \qquad (2)$$

converges to the solution u^* *and*

$$\|u^* - u^h\|_{H_0^2(\Omega)} \leq \frac{1}{m\sqrt{\lambda_1}} \|T(u_0) - g\|_{L^2(\Omega)} (\frac{M-m}{M+m})^n,$$

where $\lambda_1 > 0$ is the smallest eigenvalue of (Δ^2) on D .

Remark:

(a) Weak form of the (2) :

$$\int_\Omega D^2 z_n D^2 v = \frac{1}{2}\int_\Omega \overline{g}(E(D^2u))(D^2u.D^2v + \Delta u \Delta \text{v}) - \int_\Omega \alpha v \qquad \left(v \in H_0^2(\Omega)\right)$$

(b) From now $\alpha > 0$ is constant .

The Gradient-Fourier method :

Let $\Omega \in [0,\pi]^2$, $\lambda_{k,l}$ and $e_{k,l}(k,l = 1,2,\ldots)$ denote the eigenvalues and eigenfunctions of (Δ^2) on D, respectively :

$$\lambda_{k,l} = (k^2 + l^2)^2,\ e_{k,l}(x,y) = \frac{2}{\pi}\sin(kx)\sin(ly)\,.$$

Let us first introduce some notations. For n=0,1,… ,let

$$u_0 = 0$$

$$u_{n+1} := u_n - \frac{2}{M+m} z_n \quad (n \in \mathbb{N})$$

The Fourier method for the auxiliary equations :

Now let us focus on a single iteration step (i.e. n $\epsilon \mathbb{N}$ is fixed in the section), where (2) is replaced by

$$\begin{cases} \Delta^2 z_n = r_n \\ z_{n|\partial\Omega} = \frac{\partial^2 z_n}{\partial v^2}|_{\partial\Omega} = 0 \end{cases} \qquad (3)$$

Let $c_{k,l}(k,\, l$ = 1,2, …) be the coefficients of r_n in its Fourier series expansion, that is $c_{k,l} := \int_\Omega r_n e_{k,l}$

Let N be a positive fixed integer and

$$r_n := \sum_{k,l=1}^{N} c_{k,l} e_{k,l}$$

Define z_n as $$z_n := \sum_{k,l=1}^{N} \frac{c_{k,l}}{\lambda_{k,l}} e_{k,l}$$

A simple calculation shows that these satisfy (3) :

$$\Delta^2 z_n := \sum_{k,l=1}^{N} \frac{c_{k,l}}{\lambda_{k,l}} (\Delta^2 e_{k,l}) = \sum_{k,l=1}^{N} \frac{c_{k,l}}{\lambda_{k,l}} \lambda_{k,l} e_{k,l} = \sum_{k,l=1}^{N} c_{k,l} e_{k,l} = r_n$$

Calculation of the coefficients $c_{k,l}$:

$$c_{k,l} = \int_{\Omega} (T(u) - \alpha) e_{k,l}$$

$$= \int_{\Omega} div^2 \overline{g}(E(D^2 u))(\widetilde{D}^2 u\ ) e_{k,l} - \int_{\Omega} \alpha e_{k,l}$$

$$= \int_{\Omega} \overline{g}(\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2)(\widetilde{D}^2 u. \widetilde{D}^2 e_{k,l}) - \int_{\Omega} \alpha e_{k,l}$$

$$J = \frac{\partial^2 u}{\partial x^2} = \sum_{k,l=1}^{N} d_{k,l} \frac{\partial^2 e_{k,l}}{\partial x^2} = -\sum_{k,l=1}^{N} \frac{2}{\pi} d_{k,l} k^2 \sin(kx) \sin(ly)$$

$$E = \frac{\partial^2 u}{\partial y^2} = \sum_{k,l=1}^{N} d_{k,l} \frac{\partial^2 e_{k,l}}{\partial y^2} = -\sum_{k,l=1}^{N} \frac{2}{\pi} d_{k,l} l^2 \sin(kx) \sin(ly)$$

$$G = \frac{\partial^2 u}{\partial x \partial y} = \sum_{k,l=1}^{N} d_{k,l} \frac{\partial^2 e_{k,l}}{\partial x \partial y} = \sum_{k,l=1}^{N} \frac{2}{\pi} d_{k,l} kl \cos(kx) \cos(ly)$$

$$Q = \frac{\partial^2 e_{k,l}}{\partial x^2} = -\frac{2}{\pi} k^2 \sin(kx) \sin(ly)$$

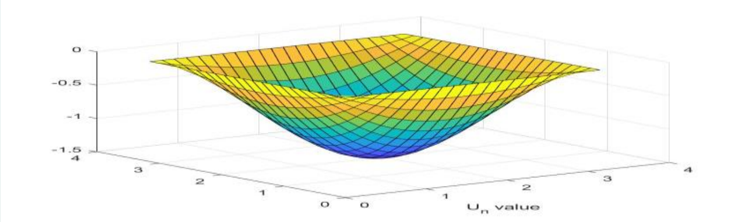
$$M = \frac{\partial^2 e_{k,l}}{\partial y^2} = -\frac{2}{\pi} l^2 \sin(kx) \sin(ly)$$

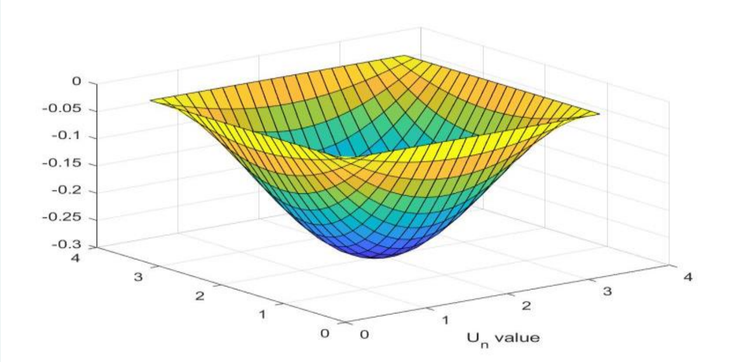
$$K = \frac{\partial^2 e_{k,l}}{\partial x \partial y} = \frac{2}{\pi} k * l \cos(kx) \cos(ly)$$

$$c_{k,l} = \int_{\Omega} \overline{g}((J)^2 + J * E + (E)^2 + (G)^2) * \left(\left(J + \frac{1}{2}E\right) * Q + G *\right.$$

Numerical experiment :

I have code it in Matlab program






```
>> gmiter4min(22,3)


errz =

   15.0368

**




errz =

   13.1337


errz =

   11.7817
```

THANK YOU